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# Evolution equations of second order with nonconvex potential and linear damping: existence via convergence of a full discretization

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**Abstract**

Global existence of solutions for a class of second-order evolution equations with damping is shown by proving convergence of a full discretization. The discretization combines a fully implicit time stepping with a Galerkin scheme. The operator acting on the zero-order term is assumed to be a potential operator where the potential may be nonconvex. A linear, symmetric operator is assumed to be acting on the first-order term. Applications arise in nonlinear viscoelasticity and elastodynamics.

*Keywords:* Evolution equation of second order, elastodynamics, nonconvex potential, full discretization, convergence
*2010 MSC:* 47J35, 65M12, 34G20, 35G25, 35Q74

## 1. Introduction

### 1.1. Problem statement

Nonlinear partial differential equations of second order in time describe a variety of problems in physical sciences and engineering. This article focuses on evolution equations of second order in time which are of the form

$$u'' + Au' + Bu = f \text{ in } (0,T), \quad u(0) = u_0, \quad u'(0) = v_0. \tag{1.1}$$

Here $A : V_A \to V_A^*$ is a linear, bounded, strongly positive and symmetric operator and $B : V_B \to V_B^*$ is a demicontinuous and bounded potential operator with potential $\phi_B$, where $V_A$ and $V_B$ are separable, reflexive Banach spaces that are continuously and densely embedded in a Hilbert space $H$. We do not assume that $V_A$ is a subspace of $V_B$ or vice versa, but $V := V_A \cap V_B$ is assumed to be continuously and densely embedded in both $V_A$ and $V_B$. Moreover, $V_A$ is assumed to be compactly embedded in $H$. The exact details will be given in Section 2. While $\phi_B$ may

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be nonconvex, we do assume that $(B + \lambda A) : V \to V^*$ is a monotone operator for some $\lambda \geq 0$. This is an Andrews–Ball-type condition (for the first use of such a condition see Andrews and Ball [2]). The potential is also assumed to be bounded from below by a constant and to be weakly coercive. Moreover, we assume that there is a Galerkin scheme for $V$ such that the $H$-orthogonal projections onto the finite dimensional subspaces are uniformly bounded as operators in $V$. This will be fulfilled in many applications.

In this setting, we prove existence of solutions to (1.1) by showing convergence (in a suitable sense) of a sequence of approximate solutions. This is, to the best knowledge of the authors, the first result in this general setting. The convergence result also implies convergence of suitable numerical schemes that are based on a conforming finite element method.

### *1.2. Illustrating examples*

For illustration, we will consider the following equations that fit into our framework:

1. Perhaps the most well-known example is the equation

$$u_{tt} - \Delta u_t - \nabla \cdot \sigma(\nabla u) = f \tag{1.2}$$

   from nonconvex elastodynamics, where the function $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ is given as the derivative of a potential $\varphi : \mathbb{R}^d \to \mathbb{R}$ and represents, e.g., the various phases in some shape-memory alloy. Examples of $\varphi$ found in the literature are usually polynomials of order strictly greater than three. Here $V_A$ and $V_B$ are Sobolev spaces corresponding to Lebesgue exponents 2 and $p \geq 2$, respectively, with $p - 1$ being the order of some polynomial that bounds the growth of $\sigma$.

2. As another example consider the equation

$$u_{tt} - \Delta u_t - \Delta\sigma(u) = f, \tag{1.3}$$

   together with appropriate initial and boundary conditions. In this equation, the functional analytic setting is somewhat unusual but the full details are given in Section 5 (as is the case for the other examples).

3. Consider finally the equation

$$u_{tt} + (-\Delta)^s u_t + \sigma(u) = f, \quad s \in (0, 1], \tag{1.4}$$

   together with appropriate initial and boundary conditions. The operator $(-\Delta)^s$ corresponds to the Laplace operator when $s = 1$ and otherwise to (a suitable definition of) the fractional Laplacian. Here $V_A$ is the Sobolev–Slobodetskii space of order $s$ with Lebesgue exponent 2 and $V_B$ is the Lebesgue space with exponent $p \geq 2$, with $p - 1$ being the order of some polynomial that bounds the growth of $\sigma$.

### *1.3. Literature overview and main result*

The main difficulties, from the point of view of applications modeling viscoelastic material, phase transformations and shape-memory alloys, are the fact that operator $B$ is not monotone, as it is given by a nonconvex potential, and that the potential should be allowed to grow at least as fast as polynomials of order four to be of practical interest. The question of modeling is subject of extensive ongoing research (see, e.g., Pego [35], Friesecke and McLeod [26], Roubíček [40], Rajagopal and Roubíček [38], and the references cited therein). The various models contain for example spatial derivatives of higher order than those in equation (1.2) (see, e.g., Arndt, Griebel and Roubíček [3] as well as Plecháč and Roubíček [36]), nonlocal operators in space (see, e.g., Ball et al. [4]), damping with memory (see, e.g., Zacher [44]) and $\sigma$ acting nonlocally in time (see, e.g., Engler [24] and Bellout, Bloom and Nečas [5]). This is one motivation for considering an abstract setting that covers, e.g., higher order spatial derivatives with operators of different order acting on the damping and the zero-order terms.

Existence and uniqueness of solutions to (1.2) has been studied extensively. In the one-dimensional case this goes back at least to Dafermos [14], Greenberg et al. [28], Andrews [1], Andrews and Ball [2], Pego [35], and Chen and Hoffmann [10]. Andrews [1] as well as Andrews and Ball [2] identified an important condition for the existence of solutions to such equations referred to as Andrews–Ball condition. We will later show that an Andrews–Ball-type condition used, e.g., by Friesecke and Dolzmann [25] can be weakened and generalized, in the abstract setting, to the monotonicity of $(B + \lambda A) : V \to V^*$ for some positive $\lambda$. Clements [11] proves existence in the multidimensional case if the operator acting on the zero-order-in-time term is monotone (that is $\sigma$ is convex). Existence and uniqueness of mild solutions has been shown in Engler [23], whereas Pecher [34] studies smooth solutions. An essential restriction in the aforementioned work is that $A$ and $B$ are differential operators of the same order.

More recent results, employing an implicit discretization in time, come from Demoulini [15], where the focus is also on the situation when there is no damping. In the presence of linear damping, Demoulini [15] proves existence of weak solutions, under the assumption that $\sigma$ is globally Lipschitz, hence allowing only quadratic growth in the potential. Friesecke and Dolzmann [25] use an implicit time discretization to show existence of solutions when the potential is not convex and $\sigma$ is locally Lipschitz continuous and growing like a polynomial of arbitrary power. However, they assume that the spatial differential operators acting on the damping and the zero-order-in-time term are both of second order. In this situation, $V_B$ is continuously embedded in $V_A$ and one can finally employ almost everywhere convergence of the gradient of the approximate solutions to deal with the nonlinear term rather than to employ Minty's monotonicity trick (see also Prohl [37] for a similar method of proof). In addition, uniqueness is shown when $\sigma$ is globally Lipschitz continuous.

For the numerical approximation of (1.2), we refer to Carstensen and Dolzmann [8] (error estimates are shown for a full discretization assuming that the solution is sufficiently regular) and Prohl [37] (convergence is shown for the same discretization as in [8] without assuming additional regularity of the weak solution). A relatively recent contribution by Demoulini, Stuart and Tzavaras [16] shows, again by employing a time discretization, that in the one-dimensional case a weak solution exists even if there is no damping (i.e., $A = 0$); in higher dimensions, the existence of Young measure valued solutions can be shown (see, e.g., Rieger [39] as well as Carstensen and Rieger [9] for the approximation of such solutions).

Using a Galerkin method, Gajewski, Gröger and Zacharias [27, Kapitel VII, Satz 1.2] show existence and uniqueness for the abstract problem (1.1) in the situation when $V_A = V_B$, which corresponds to the case when $\sigma$ has at most quadratic growth. Moreover, the operator $B$ is required to be Lipschitz continuous. The abstract setting studied in Roubíček [41, Chapter 1, Section 11.3] is again restricted to the case $V_A = V_B$ but allows $B$ to be a semi-coercive and pseudomonotone operator. The restriction to the case $V_A = V_B$ is a severe restriction since the assumptions on $A$ imply that $V_A = V_B$ is a Hilbert space. The class of nonlinear operators $B$ is, therefore, quite restricted.

Another motivation for studying the setting in this paper is to complement results on nonlinear evolution equations of second order that have been obtained recently. If the operator $B : V_B \to V_B^*$, which is the operator acting on the zero-order term, is linear, bounded, strongly positive and symmetric and $A : V_A \to V_A^*$ is hemicontinuous, coercive, monotone and satisfies a growth condition then a unique solution exists without any requirement on continuous embeddings between $V_A$ and $V_B$. This is due to Lions and Strauss [32]. In this setting, Emmrich and Thalhammer [21] have proved weak convergence of time discretizations under the assumption that $V_A$ is continuously embedded in $V_B$. Later this has been extended, in Emmrich and Thalhammer [22], where existence of solutions and weak convergence of fully discrete approximations has been proved in the case when nonmonotone perturbations are added to $A$ and $B$ and even if $V_A$ is not continuously embedded in $V_B$. The convergence results have subsequently been extended in Emmrich and Šiška [20].

The main result of this paper is the proof of existence of solutions to the evolution equation (1.1) in the case when $B$ is given by a nonconvex potential with the only restriction on growth being that it maps bounded sets into bounded sets and is bounded from below by a constant. Thus the potential which defines $B$ may grow faster than polynomials of an arbitrary order. We do not need to assume that $V_B$ is continuously embedded in $V_A$ or vice versa. We also prove (strong) convergence of a full discretization, which provides a theoretical substantiation of the numerical approximation by combining the implicit time stepping scheme with a conforming finite element method.

This extends what is known due to Friesecke and Dolzmann [25] and due to Prohl [37] for the example (1.2) since we do not need to assume that the differential operators acting on the zero-order-in-time and first-order-in-time terms are second-order differential operators. Our proof differs from that in Prohl [37]. There, the monotonicity of $(B + \lambda A) : V \to V^*$ is only used to show strong convergence of a subsequence of the approximating sequence in the appropriate space, but then almost everywhere convergence of the gradient of the approximate solution is used to identify the limit in the nonlinear term. This only works when the operators are both second-order differential operators in divergence form. Instead, in this paper, the monotonicity of $(B + \lambda A) : V \to V^*$ is used again at the final step to identify the limit. Compared with Demoulini [15], we also treat the situation when the nonconvex potential grows faster than a second-order polynomial.

We only consider operators that are constant in time. However, provided all the assumptions are satisfied uniformly in time, it should be possible to extend the results to operators that are not constant in time. Incorporating nonmonotone (strongly continuous) perturbations will be left for future work.

### *1.4. Organization of the paper*

The paper is organized as follows. In Section 2, we give the precise assumptions on the function spaces and operators involved and we introduce the full discretization. In Section 3, we show that the fully discrete problem has a unique solution and we prove a priori estimates for this solution. In Section 4, we state the main result of this paper: the existence of solutions to (1.1). This will be proved by taking the limit of the fully discrete problem with respect to the discretization parameters. In Section 5, we return to the applications mentioned in the introduction. In an appendix, we finally provide an integration-by-parts formula, which is essential to proving the main result of this paper.

## 2. Spaces, operators, assumptions and the full discretization

This section provides the exact function space setting, the assumptions on the operators and the approximating scheme that will be used to prove existence of solutions to problem (1.1).

### 2.1. Function space setting

Let $(V_A, \|\cdot\|_{V_A})$ be a real, reflexive and separable Banach space that is continuously and densely embedded in a real Hilbert space $(H, (\cdot,\cdot), |\cdot|)$ such that $V_A \subseteq H \subseteq V_A^*$ form a Gelfand triple. Let $(V_B, \|\cdot\|_{V_B})$ be a real, reflexive and separable Banach space such that $V_B \subseteq H \subseteq V_B^*$ again form a Gelfand triple. Furthermore, let $V := V_A \cap V_B$, endow it with the norm $\|\cdot\|_V = \|\cdot\|_{V_A} + \|\cdot\|_{V_B}$ and assume that $V$ is separable and dense in both the spaces $V_A$ and $V_B$. The dual $V^*$ of $V$ can be identified with $V_A^* + V_B^*$ and is a Banach space when equipped with the norm

$$\|g\|_{V^*} = \inf\left\{\max\left(\|g_A\|_{V_A^*}, \|g_B\|_{V_B^*}\right) : g = g_A + g_B, g_A \in V_A^*, g_B \in V_B^*\right\},$$

see, e.g., Gajewski, Gröger and Zacharias [27, Kapitel I, Satz 5.13]. Since $V_A$ and $V_B$ are both assumed to be reflexive, $V$ is also reflexive. The duality pairing between $g = g_A + g_B \in V^* = V_A^* + V_B^*$ and $w \in V$ is given by

$$\langle g, w\rangle = \langle g_A, w\rangle_{V_A^*\times V_A} + \langle g_B, w\rangle_{V_B^*\times V_B}.$$

Thus we have the following scale of spaces:

$$V_A \cap V_B = V \subseteq V_C \subseteq H = H^* \subseteq V_C^* \subseteq V^* = V_A^* + V_B^*, \quad C \in \{A, B\},$$

with continuous and dense embeddings.

By $L^r(0,T;X)$ with $r \in [1,\infty]$, we denote the usual spaces of Bochner integrable (for $r = \infty$ Bochner measurable and essentially bounded) abstract functions mapping $[0,T]$ into a (reflexive) Banach space $X$, equipped with the standard norm denoted by $\|\cdot\|_{L^r(0,T;X)}$.

We will always assume that $p \in [2,\infty)$ and set $p^* = p/(p-1)$. The duality pairing between $L^p(0,T;V) \ni w$ and $(L^p(0,T;V))^* = L^{p^*}(0,T;V^*) = L^{p^*}(0,T;V_A^*) + L^{p^*}(0,T;V_B^*) \ni g = g_A + g_B$ is given by

$$\langle g, w\rangle = \int_0^T \langle g(t), w(t)\rangle_{V^*\times V} dt = \int_0^T \langle g_A(t), w(t)\rangle_{V_A^*\times V_A} dt + \int_0^T \langle g_B(t), w(t)\rangle_{V_B^*\times V_B} dt\,.$$

For more details on Bochner–Lebesgue spaces, we refer to Diestel and Uhl [17].

Let $X$ be again a Banach space. By $\mathscr{AC}([0,T],X)$, $\mathscr{C}([0,T],X)$ and $\mathscr{C}_w([0,T],X)$, we denote the spaces of absolutely continuous, continuous and weakly continuous functions mapping $[0,T]$ into $X$, respectively. Let $w'$ and $w''$ denote the first and second time derivative of the abstract function $w = w(t)$ in the distributional sense. By $H^1(0,T;X)$, we denote the Banach space of functions $w \in L^2(0,T;X)$ with $w' \in L^2(0,T;X)$, equipped with the standard norm. Note that $H^1(0,T;X)$ is continuously embedded in $\mathscr{C}([0,T],X)$ and that $H^1(0,T;X) \subseteq \mathscr{AC}([0,T],X)$. The space of continuously differentiable functions mapping $[0,T]$ into $X$ is denoted by $\mathscr{C}^1([0,T],X)$.

Finally, let $\mathscr{C}_c^\infty(0,T)$ be the space of infinitely many times differentiable real functions with compact support in $(0,T)$. By $c$, we denote a generic positive constant.

### 2.2. Assumptions on the operators

In this subsection, detailed assumptions on the operators will be given.

**Assumption A.** *Let $A : V_A \to V_A^*$ be linear, symmetric, bounded and strongly positive. In particular there are constants $c_A > 0$ and $\mu_A > 0$ so that for all $w, z \in V_A$*

$$\langle Aw, z\rangle \leq c_A \|w\|_{V_A} \|z\|_{V_A} \quad \textit{and} \quad \langle Aw, w\rangle \geq \mu_A \|w\|_{V_A}^2 . \tag{2.1}$$

So $A : V_A \to V_A^*$ defines an inner product on $V_A$. We denote the norm induced by this inner product by $\|\cdot\|_A := \langle A\cdot, \cdot\rangle^{1/2}$ and note that this norm is equivalent to $\|\cdot\|_{V_A}$. Furthermore, we can define the potential $\phi_A(w) = \frac{1}{2}\langle Aw, w\rangle$. Then the Gâteaux derivative of $\phi_A : V_A \to \mathbb{R}$ exists and $\phi_A' = A$.

Let us note that the linear, bounded operator $A : V_A \to V_A^*$ extends to a linear, bounded operator mapping $L^2(0, T; V_A)$ into $L^2(0, T; V_A^*)$ via $(Aw)(t) := Aw(t)$ for $w \in L^2(0, T; V_A)$.

**Assumption B.** *Let $B : V_B \to V_B^*$ be a bounded and demicontinuous potential operator with the potential $\phi_B : V_B \to \mathbb{R}$ such that the potential is weakly coercive and bounded from below by a constant.*

So we assume that the Gâteaux derivative $\phi_B' : V_B \to V_B^*$ of $\phi_B$ exists and $B = \phi_B'$. Saying that $B : V_B \to V_B^*$ is bounded means that $B$ maps bounded subsets of $V_B$ into bounded subsets of $V_B^*$. Demicontinuity of $B : V_B \to V_B^*$ means that for any $z \in V_B$ the mapping $w \mapsto \langle Bw, z\rangle$ is continuous as a mapping of $V_B$ into $\mathbb{R}$. Weak coercivity of $\phi_B$ means that if $\|w\|_{V_B} \to \infty$ then $\phi_B(w) \to \infty$ and boundedness from below means that there is $c_B > 0$ such that $\phi_B(w) \geq -c_B$ for all $w \in V_B$.

Note that the demicontinuity of the operator $B : V_B \to V_B^*$ implies its local boundedness. Let us further remark that if $\phi_B : V_B \to \mathbb{R}$ is Gâteaux differentiable with $B = \phi_B' : V_B \to V_B^*$ bounded then $\phi_B : V_B \to \mathbb{R}$ is locally Lipschitz continuous. Finally, if the potential $\phi_B : V_B \to \mathbb{R}$ would, in addition to the weak coercivity, also be weakly sequentially lower semicontinuous then this would already imply that it is bounded from below by a constant. For more details, we refer to Gajewski, Gröger and Zacharias [27, Kapitel III].

It can be shown that $B : V_B \to V_B^*$ extends to an operator mapping $L^\infty(0, T; V_B)$ into $L^\infty(0, T; V_B^*)$ via $(Bw)(t) := Bw(t)$ for $w \in L^\infty(0, T; V_B)$: The demicontinuity of $B : V_B \to V_B^*$ together with Pettis' theorem (see, e.g., Diestel and Uhl [17, Chapter II, Section 1, Theorem 2]) implies that Bochner measurable functions with values in $V_B$ are mapped into Bochner measurable functions with values in $V_B^*$. The boundedness of $B : V_B \to V_B^*$ then shows that an essentially bounded function with values in $V_B$ is mapped into an essentially bounded function with values in $V_B^*$. Moreover, the mapping $B : L^\infty(0, T; V_B) \to L^\infty(0, T; V_B^*)$ is bounded.

We know that for any $w \in V_B$

$$\phi_B(w) = \phi_B(0) + \int_0^1 \langle B(tw), w\rangle dt ,$$

see, e.g., Roubíček [41, Chapter 4, Section 1] or Gajewski, Gröger and Zacharias [27, Kapitel III, Bemerkung 4.1]. Hence it immediately follows that if $B : V_B \to V_B^*$ maps bounded sets into bounded sets then $\phi_B$ maps bounded sets in $V_B$ into bounded sets in $\mathbb{R}$.

Finally, we need the following relation between $A$ and $B$, which is a condition of Andrews–Ball type.

**Assumption AB.** *Let there be $\lambda \geq 0$ such that $(B + \lambda A) : V \to V^*$ is monotone, i.e., for all $w, z \in V$*

$$\langle Bw - Bz, w - z\rangle \geq -\lambda \|w - z\|_A^2. \tag{2.2}$$

Here, the operator $B$ is only considered as an operator mapping $V$ into $V_B^*$ and the operator $A$ is only considered as an operator mapping $V$ into $V_A^*$. As $V^*$ is identified with $V_A^* + V_B^*$, the linear combination of $A$ and $B$ can be considered as an operator mapping $V$ into $V^*$.

Consider for a moment the specific situation where $A$ is the Laplacian, in the weak sense with homogenous Dirichlet boundary conditions, and $B$ is given by the mapping $u \mapsto -\nabla \cdot \sigma(\nabla u)$, in the weak sense with homogenous Dirichlet boundary conditions, while $\sigma$ arises as the derivative of some given potential. Then Andrews [1] as well as Andrews and Ball [2] use, in particular, the assumption that for some $R > 0$

$$(\sigma(x) - \sigma(y)) \cdot (x - y) > 0, \quad \text{whenever } |x - y| \geq R, \quad x, y \in \mathbb{R}^d \tag{2.3}$$

in order to prove global existence of a corresponding one-dimensional problem. It can be shown that the Andrews–Ball-type condition

$$(\sigma(x) - \sigma(y)) \cdot (x - y) \geq 0, \quad \text{whenever } |x|, |y| \geq R, \quad x, y \in \mathbb{R}^d, \tag{2.4}$$

which was later employed in, e.g., Friesecke and Dolzmann [25], together with local Lipschitz continuity of $\sigma$, implies that for some $\lambda > 0$

$$(\sigma(x) - \sigma(y)) \cdot (x - y) \geq -\lambda |x - y|^2, \quad x, y \in \mathbb{R}^d. \tag{2.5}$$

Indeed, if both $x$ and $y$ are such that $|x|, |y| \geq R$ then the estimate follows from (2.4). If $x$ and $y$ are both in the closed ball of radius $R$ then the function $\sigma$, restricted to this ball, is globally Lipschitz continuous with some constant $L_R$ and we simply choose $\lambda \geq L_R$. The last remaining case is when $|x| > R$ but $|y| < R$. In this case, consider $z \in \mathbb{R}^d$ such that $|z| = R$ and $z$ lies on the line segment between $x$ and $y$. That is, $z = x + \theta(y - x)$ for some $\theta \in (0, 1)$. We find

$$\begin{aligned}(\sigma(x) - \sigma(y)) \cdot (x - y) &= (\sigma(x) - \sigma(z)) \cdot (x - y) + (\sigma(z) - \sigma(y)) \cdot (x - y) \\ &= \frac{1}{\theta}(\sigma(x) - \sigma(z)) \cdot (x - z) + (\sigma(z) - \sigma(y)) \cdot (x - y) \\ &\geq 0 - L_R |z - y||x - y| = -L_R(1 - \theta) |x - y|^2,\end{aligned}$$

with the estimate coming from (2.4) for $|x|, |z| \geq R$ and from the Lipschitz continuity of $\sigma$ when restricted to the closed ball of radius $R$. This shows that Assumption AB generalizes the Andrews–Ball-type condition (2.4) to the abstract setting. The connection between the original Andrews–Ball condition (2.3) and (2.4) or (2.5) is not immediate. The condition (2.5) is the one that is used in, e.g., Prohl [37] and Rieger [39].

To conclude the discussion about the assumptions placed on the operators $A$ and $B$, we make the following simple observation.

**Lemma 2.1.** *Let the potential $\phi$ be defined as $\phi(w) := \phi_B(w) + \lambda\phi_A(w)$ for any $w \in V$. Let Assumptions* A *and* B *hold. Then $\phi' = B + \lambda A$ and $(B + \lambda A) : V \to V^*$ is monotone if and only if for all $w, z \in V$*

$$\langle \phi_B'(w), w - z\rangle \geq \phi_B(w) - \phi_B(z) - \lambda\phi_A(w - z).$$

*Proof.* Due to Gajewski, Gröger and Zacharias [27, Kapitel III, Lemma 4.10], we know that $(B + \lambda A) : V \to V^*$ is monotone if and only if for all $w, z \in V$

$$\langle \phi'(w), w - z\rangle \geq \phi(w) - \phi(z).$$

Simply by rearranging the terms in the inequality, this is equivalent to

$$\langle \phi_B'(w), w - z\rangle \geq \phi_B(w) - \phi_B(z) + \lambda \left(\phi_A(w) - \phi_A(z) - \langle Aw, w - z\rangle\right).$$

Observe that

$$\phi_A(w) - \phi_A(z) - \langle Aw, w - z\rangle = -\frac{1}{2}\langle Aw - Az, w - z\rangle = -\phi_A(w - z).$$

This proves the assertion. $\square$

*2.3. Full discretization*

The numerical scheme will be derived from the first order system

$$\begin{cases} u' - v = 0, \\ v' + Av + Bu = f \ \text{ in } \ (0, T), \quad u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{2.6}$$

which is formally equivalent to (1.1).

Application of the implicit Euler scheme to both the first and second equation will give us our temporal discretization scheme. For given $N \in \mathbb{N}$ let $\tau := T/N$. Let $\{V_m\}_{m\in\mathbb{N}}$ be a Galerkin scheme for $V$ (recall that $V$ is assumed to be separable, hence a Galerkin basis exists; without loss of generality, we assume that $V_k \subseteq V_m$ for $k \leq m$ and that the dimension of $V_m$ is $m$). Let $u^0$ and $v^0$ in $V_m$ be some approximations of the initial data $u_0$ and $v_0$, respectively. Let $\{f^n\}_{n=1}^N \subset V_A^*$ be some approximation of the right-hand side. We look for $u^n \approx u(t_n)$, $v^n \approx v(t_n)$ with $u^n$, $v^n \in V_m$ such that for $n = 1, \ldots, N$

$$\begin{cases} \dfrac{1}{\tau}(u^n - u^{n-1}, \varphi) - (v^n, \varphi) = 0 \quad \forall \varphi \in V_m, \\ \dfrac{1}{\tau}\left(v^n - v^{n-1}, \varphi\right) + \langle Av^n, \varphi\rangle + \langle Bu^n, \varphi\rangle = \langle f^n, \varphi\rangle \quad \forall \varphi \in V_m. \end{cases} \tag{2.7}$$

Let us mention that $v^n$ as well as $(u^n - u^{n-1})/\tau$ are in $V_m$. The first equation thus implies equality of $v^n$ and $(u^n - u^{n-1})/\tau$ in $H$ since one may take $\varphi = v^n - (u^n - u^{n-1})/\tau$, which shows that $|v^n - (u^n - u^{n-1})/\tau| = 0$.

Solving the first equation for $v^n$ and substituting into the second equation in (2.7), we obtain the equivalent formulation

$$\left(\frac{u^n - 2u^{n-1} + u^{n-2}}{\tau^2}, \varphi\right) + \left\langle A\frac{u^n - u^{n-1}}{\tau}, \varphi\right\rangle + \langle Bu^n, \varphi\rangle = \langle f^n, \varphi\rangle \quad \forall \varphi \in V_m,$$

with $u^0$ and $u^{-1} := u^0 - \tau v^0$ given. We remark that the scheme is different from the explicit-implicit Euler scheme (also known as the Störmer–Verlet or leap-frog scheme) used in Emmrich and Thalhammer [22]. In the present setting it does not seem possible to obtain the required a priori estimates for the explicit-implicit Euler scheme.

It is also worth noting that (2.6) can be treated as a Volterra integro-differential equation. Indeed, let $(Kv)(t) := \int_0^t v(s)ds$. Then (2.6) corresponds to

$$v' + Av + B(u_0 + Kv) = f \text{ in } (0,T), \quad v(0) = v_0.$$

Similarly (2.7) can be reformulated as

$$\frac{1}{\tau}\left(v^n - v^{n-1}, \varphi\right) + \langle Av^n, \varphi\rangle + \left\langle B\left(u^0 + \tau\sum_{k=1}^{n} v^k\right), \varphi\right\rangle = \langle f^n, \varphi\rangle \quad \forall\varphi \in V_m,$$

for $n = 1, \ldots, N$.

## 3. Properties of the full discretization

In this section, we show that the discrete problem (2.7) has, under the right assumptions, a unique solution. Moreover, we derive a priori estimates which will be essential for proving convergence of a sequence of approximate solutions.

### *3.1. Existence and uniqueness for the discrete problem*

Existence of solutions to the discrete problem will be proved by applying the following lemma.

**Lemma 3.1.** *Let $\boldsymbol{h} : \mathbb{R}^m \to \mathbb{R}^m$ be continuous. If there is $R > 0$ such that $\boldsymbol{h}(\boldsymbol{v}) \cdot \boldsymbol{v} \geq 0$ whenever $\|\boldsymbol{v}\|_{\mathbb{R}^m} = R$ then there exists $\bar{\boldsymbol{v}}$ satisfying $\|\bar{\boldsymbol{v}}\|_{\mathbb{R}^m} \leq R$ and $\boldsymbol{h}(\bar{\boldsymbol{v}}) = 0$.*

*Proof.* The lemma is proved by contradiction from Brouwer's fixed point theorem (see, e.g., Gajewski, Gröger and Zacharias [27, Kapitel III, Lemma 2.1]). □

We are now ready to prove existence of solutions to the full discretization.

**Theorem 3.2** (Existence for discrete problem)**.** *Let Assumptions* A, B *and* AB *hold and let, if $\lambda \neq 0$, the time step be sufficiently small such that $\tau \leq \mu_A/(\lambda c_A)$. Then, given $u^0, v^0 \in V_m$ and $\{f^n\}_{n=1}^N \subset V_A^*$, the fully discrete problem* (2.7) *has a solution $\{u^n\}_{n=1}^N$, $\{v^n\}_{n=1}^N \subset V_m$.*

*Proof.* We prove the existence step by step. Assume that we already know $\{u^j\}_{j=0}^{n-1} \subset V_m$, $\{v^j\}_{j=0}^{n-1} \subset V_m$. We would like to find $u^n$, $v^n$ satisfying (2.7). Let $\{\varphi_i\}_{i=1}^m$ be a basis for $V_m$. There is a one-to-one correspondence between any $w \in V_m$ and $\boldsymbol{w} = (w_1, \ldots, w_m)^T \in \mathbb{R}^m$ given by

$$w = \sum_{i=1}^{m} w_i\varphi_i,$$

where we assume, without loss of generality, that the dimension of $V_m$ is $m$. For an arbitrary $v \in V_m$ and hence for the associated $\boldsymbol{v} = (v_1, \ldots, v_m)^T \in \mathbb{R}^m$, define $\boldsymbol{h} : \mathbb{R}^m \to \mathbb{R}^m$, componentwise for $j = 1, \ldots, m$, as

$$h(\boldsymbol{v})_j := \frac{1}{\tau}(v - v^{n-1}, \varphi_j) + \langle Av, \varphi_j\rangle + \langle B(u^{n-1} + \tau v), \varphi_j\rangle - \langle f^n, \varphi_j\rangle.$$

Then, showing that (2.7) has a solution amounts to showing that there is some $\boldsymbol{v} \in \mathbb{R}^m$ such that $\boldsymbol{h}(\boldsymbol{v}) = \boldsymbol{0}$. To that end, we would like to apply Lemma 3.1. Let $\|\cdot\|_{\mathbb{R}^m} := \|\cdot\|_{V_A}$. Observe that

$$\boldsymbol{h}(\boldsymbol{v}) \cdot \boldsymbol{v} = \frac{1}{\tau}(v - v^{n-1}, v) + \langle Av, v\rangle + \langle B(u^{n-1} + \tau v), v\rangle - \langle f^n, v\rangle.$$

Furthermore, due to Lemma 2.1, we have

$$\langle B(u^{n-1}+\tau v), v\rangle \geq \frac{1}{\tau}\left(\phi_B(u^{n-1}+\tau v) - \phi_B(u^{n-1}) - \lambda\phi_A(\tau v)\right).$$

Hence, using (2.1), using the lower bound for $\phi_B$ and $V_A \hookrightarrow H$, we get

$$\boldsymbol{h}(\boldsymbol{v})\cdot\boldsymbol{v} \geq \mu_A\|v\|_{V_A}^2 - \frac{1}{2}\lambda\tau c_A\|v\|_{V_A}^2 - \frac{1}{\tau}\phi_B(u^{n-1}) - \|v\|_{V_A}\left(\frac{c}{\tau}|v^{n-1}| + \|f^n\|_{V_A^*}\right) - \frac{c_B}{\tau}.$$

As we are assuming that $\mu_A \geq \lambda\tau c_A$, we get that

$$\boldsymbol{h}(\boldsymbol{v})\cdot\boldsymbol{v} \geq \|v\|_{V_A}\left(\frac{\mu_A}{2}\|v\|_{V_A} - \frac{c}{\tau}|v^{n-1}| - \|f^n\|_{V_A^*}\right) - \frac{1}{\tau}\phi_B(u^{n-1}) - \frac{c_B}{\tau}.$$

From this we can see that $R > 0$ can be chosen sufficiently large so that if $\|\boldsymbol{v}\|_{\mathbb{R}^m} = \|v\|_{V_A} = R$ then $\boldsymbol{h}(\boldsymbol{v})\cdot\boldsymbol{v} \geq 0$. Finally, the demicontinuity of $B : V_B \to V_B^*$ and linearity and boundedness of $A : V_A \to V_A^*$ imply the continuity of $\boldsymbol{h}$. Thus, by Lemma 3.1, there is a solution to $\boldsymbol{h}(\boldsymbol{v}) = \boldsymbol{0}$, which corresponds to $v^n$. Step by step, we get a solution to (2.7). □

The monotonicity of the operator $(B+\lambda A) : V \to V^*$ for some $\lambda \geq 0$, that is the generalized Andrews–Ball-type condition, is crucial in proving uniqueness of solutions to the numerical scheme.

**Theorem 3.3** (Uniqueness for discrete problem)**.** *Let Assumption* AB *be satisfied and let, if $\lambda \neq 0$, the time step be sufficiently small such that $\tau \leq 1/\lambda$. Then the solution to* (2.7) *is unique.*

*Proof.* We will prove the uniqueness step by step. That is, we will show that if two solutions $\{u_1^k\}_{k=0}^N$ and $\{u_2^k\}_{k=0}^N$ to (2.7) with identical right-hand side coincide up to $k = n-1$ then $u_1^n = u_2^n$. Note that $v_1^{n-1} = v_2^{n-1}$. Let

$$w^n := v_1^n - v_2^n = \frac{u_1^n - u_2^n}{\tau}.$$

Now we subtract the second equation in (2.7) for $v_2^n$ from the one for $v_1^n$ and test with $w^n$ to obtain

$$\frac{1}{\tau}|w^n|^2 + \langle Aw^n, w^n\rangle + \left\langle Bu_1^n - Bu_2^n, w^n\right\rangle = 0.$$

Hence

$$\frac{1}{\tau}|w^n|^2 + \frac{1}{\tau}\left\langle\left(B + \frac{1}{\tau}A\right)u_1^n - \left(B + \frac{1}{\tau}A\right)u_2^n, u_1^n - u_2^n\right\rangle = 0.$$

Finally, the monotonicity of $(B+\lambda A) : V \to V^*$ together with $\lambda\tau \leq 1$ gives $|w^n|^2 \leq 0$. Hence $v_1^n = v_2^n$ as well as $u_1^n = u_2^n$. □

Note that the time step restriction in Theorem 3.2 implies the one in Theorem 3.3 since $\mu_A \leq c_A$.

*3.2. A priori estimates for the discrete problem*

The first a priori estimate is proved by testing with $v^n$ in the second equation in (2.7) and using, in particular, the generalized Andrews–Ball-type condition (Assumption AB).

**Theorem 3.4** (Discrete a priori estimate I)**.** *Let Assumptions* A, B *and* AB *hold and let, if $\lambda \neq 0$, $\tau \leq \mu_A/(2\lambda c_A)$. Let $\{u^n\}_{n=0}^N \subset V_m$, $\{v^n\}_{n=0}^N \subset V_m$ be the solution of* (2.7)*. Then for any $n = 1, \dots, N$*

$$
\begin{aligned}
|v^n|^2 + \sum_{j=1}^{n} |v^j - v^{j-1}|^2 + \frac{\mu_A}{2}\tau \sum_{j=1}^{n} \|v^j\|_{V_A}^2 + 2\phi_B(u^n)& \\
\leq |v^0|^2 + 2\phi_B(u^0) + \frac{\tau}{\mu_A} \sum_{j=1}^{n} \|f^j\|_{V_A^*}^2&
\end{aligned}
\tag{3.1a}
$$

*as well as*

$$
\|u^n - u^0\|_{V_A}^2 \leq \frac{2T}{\mu_A}\left(|v^0|^2 + 2\phi_B(u^0) + \frac{\tau}{\mu_A}\sum_{j=1}^{n}\|f^j\|_{V_A^*}^2\right). \tag{3.1b}
$$

*Proof.* We test the second equation of (2.7) with $v^n$ and use the algebraic relation

$$
(a-b)a = \frac{1}{2}\left(a^2 - b^2 + (a-b)^2\right), \quad a, b \in \mathbb{R}\,.
$$

To obtain the estimates, we note that $v^n = (u^n - u^{n-1})/\tau$ and hence, due to Lemma 2.1,

$$
\langle Bu^n, v^n\rangle = \frac{1}{\tau}\langle Bu^n, u^n - u^{n-1}\rangle \geq \frac{1}{\tau}\left(\phi_B(u^n) - \phi_B(u^{n-1}) - \lambda\phi_A(u^n - u^{n-1})\right).
$$

Strong positivity of $A : V_A \to V_A^*$ together with the above algebraic relation and Young's inequality yields for $n = j$

$$
\begin{aligned}
&\frac{1}{2\tau}\left(|v^j|^2 - |v^{j-1}|^2 + |v^j - v^{j-1}|^2\right) + \mu_A\|v^j\|_{V_A}^2 \\
&\quad + \frac{1}{\tau}\phi_B(u^j) - \frac{1}{\tau}\phi_B(u^{j-1}) - \frac{\lambda}{\tau}\phi_A(u^j - u^{j-1}) \leq \frac{1}{2\mu_A}\|f^j\|_{V_A^*}^2 + \frac{\mu_A}{2}\|v^j\|_{V_A}^2.
\end{aligned}
$$

Recall that $\phi_A(w) = \frac{1}{2}\langle Aw, w\rangle \leq \frac{1}{2}c_A\|w\|_{V_A}^2$ for all $w \in V_A$. We multiply the above equation by $2\tau$ and sum from $j = 1$ to $n$. Hence we obtain

$$
\begin{aligned}
&|v^n|^2 + \sum_{j=1}^{n}|v^j - v^{j-1}|^2 + \mu_A\tau\sum_{j=1}^{n}\|v^j\|_{V_A}^2 + 2\phi_B(u^n) \\
&\qquad \leq |v^0|^2 + 2\phi_B(u^0) + \lambda c_A\sum_{j=1}^{n}\|u^j - u^{j-1}\|_{V_A}^2 + \frac{\tau}{\mu_A}\sum_{j=1}^{n}\|f^j\|_{V_A^*}^2.
\end{aligned}
$$

At this point, we note that

$$
\sum_{j=1}^{n}\|u^j - u^{j-1}\|_{V_A}^2 = \tau^2\sum_{j=1}^{n}\|v^j\|_{V_A}^2.
$$

But, due to our assumption on $\tau$, we have $\lambda c_A \tau \leq \mu_A/2$ and hence

$$|v^n|^2 + \sum_{j=1}^{n} |v^j - v^{j-1}|^2 + \frac{\mu_A}{2}\tau \sum_{j=1}^{n} \|v^j\|_{V_A}^2 + 2\phi_B(u^n)$$
$$\leq |v^0|^2 + 2\phi_B(u^0) + \frac{\tau}{\mu_A} \sum_{j=1}^{n} \|f^j\|_{V_A^*}^2.$$

This completes the proof of the first statement of the theorem. To prove the second statement, observe that

$$u^n - u^0 = \tau \sum_{j=1}^{n} v^j.$$

Hence, using Hölder's inequality,

$$\|u^n - u_0\|_{V_A}^2 \leq T \left( \tau \sum_{j=1}^{n} \|v^j\|_{V_A}^2 \right).$$

Noticing that the first part of the theorem gives us an estimate for the right-hand side of this inequality completes the proof. □

**Theorem 3.5** (Discrete a priori estimate II)**.** *Let Assumptions* A, B *and* AB *hold and let, if* $\lambda \neq 0$, $\tau \leq \mu_A/(2\lambda c_A)$*. By* $P_m$ *denote the* $H$*-orthogonal projection onto* $V_m$*. Let*

$$\|P_m\|_{V \leftarrow V} := \sup_{w \in V \setminus \{0\}} \frac{\|P_m w\|_V}{\|w\|_V}.$$

*Let* $\{u^n\}_{n=0}^N \subset V_m$*,* $\{v^n\}_{n=0}^N \subset V_m$ *be the solution to* (2.7)*. Then*

$$\tau \sum_{n=1}^{N} \left\| \frac{v^n - v^{n-1}}{\tau} \right\|_{V^*}^2 \leq c \, \|P_m\|_{V \leftarrow V}^2 \left( |v^0|^2 + \phi_B(u^0) + \tau \sum_{n=1}^{N} \|f^n\|_{V_A^*}^2 + \max_{n=1,\dots,N} \|Bu^n\|_{V_B^*}^2 \right).$$

*Proof.* Since $v^n$ and $v^{n-1}$ are in $V_m \subseteq V \subseteq H$ and thanks to the $H$-orthogonality of the projection $P_m$, we have

$$\left\| \frac{v^n - v^{n-1}}{\tau} \right\|_{V^*} = \sup_{v \in V \setminus \{0\}} \frac{1}{\|v\|_V} \left\langle \frac{v^n - v^{n-1}}{\tau}, v \right\rangle$$
$$= \sup_{v \in V \setminus \{0\}} \frac{1}{\|v\|_V} \frac{\|P_m v\|_V}{\|P_m v\|_V} \left( \frac{v^n - v^{n-1}}{\tau}, P_m v \right).$$

Since $\{v^n\}_{n=0}^N$ satisfies the second equation in (2.7) and $P_m v \in V_m$, we get

$$\left\| \frac{v^n - v^{n-1}}{\tau} \right\|_{V^*} = \sup_{v \in V \setminus \{0\}} \frac{\|P_m v\|_V}{\|v\|_V} \frac{\langle f^n, P_m v \rangle - \langle A v^n, P_m v \rangle - \langle B u^n, P_m v \rangle}{\|P_m v\|_V}.$$

Using Assumptions A and B, together with the observation that $\|\cdot\|_{V_A} \leq \|\cdot\|_V$ and $\|\cdot\|_{V_B} \leq \|\cdot\|_V$, we arrive at

$$\left\| \frac{v^n - v^{n-1}}{\tau} \right\|_{V^*} \leq \|P_m\|_{V \leftarrow V} \left( \|f^n\|_{V_A^*} + c_A \|v^n\|_{V_A} + \|Bu^n\|_{V_B^*} \right).$$

Squaring the above inequality, applying Young's inequality, multiplying by $\tau$ and summing up from $n = 1$ to $N$ gives

$$\tau \sum_{n=1}^{N} \left\| \frac{v^n - v^{n-1}}{\tau} \right\|_{V^*}^2 \le c \, \|P_m\|_{V \leftarrow V}^2 \left( \tau \sum_{n=1}^{N} \|f^n\|_{V_A^*}^2 + \tau \sum_{n=1}^{N} \|v^n\|_{V_A}^2 + \max_{n=1,\ldots,N} \|Bu^n\|_{V_B^*}^2 \right).$$

The claim now follows from the previous a priori estimate in Theorem 3.4. $\square$

We note that the term with $\|Bu^n\|_{V_B^*}^2$ can be handled later due to the first a priori estimate (3.1a) and since $\phi_B : V_B \to \mathbb{R}$ is assumed to be weakly coercive and $B : V_B \to V_B^*$ is assumed to be bounded.

## 4. Convergence towards a weak solution

### *4.1. Assumptions and statement of the existence result*

Consider some sequence $\{(N_\ell, m_\ell)\}_{\ell \in \mathbb{N}}$ such that $N_\ell \to \infty$ and $m_\ell \to \infty$ as $\ell \to \infty$. Let $\tau_\ell := T/N_\ell$. We introduce the following uniform time grid on $[0, T]$:

$$t_0 = 0 < \cdots < t_{n,\ell} = n\tau_\ell < \cdots < t_{N_\ell} = N_\ell \tau_\ell = T.$$

**Assumption P** (Projection)**.** *There is $c > 0$ such that $\|P_{m_\ell}\|_{V \leftarrow V} \le c$ for all $\ell \in \mathbb{N}$, where $P_{m_\ell}$ is the $H$-orthogonal projection onto $V_{m_\ell} \subseteq V \subseteq H$.*

To the best knowledge of the authors, it is an open question under which assumptions on $V$ and $H$ a Galerkin scheme for $V$ exists such that Assumption P holds. However, regarding standard applications, Assumption P is satisfied. Note that if the projection is stable as a linear and bounded operator in $V_A$ as well as in $V_B$ then it is also stable in $V$. The stability of the $L^2(\Omega)$-orthogonal projection onto suitable finite element spaces $V_m$ as an operator in the standard Sobolev space $W^{1,p}(\Omega)$ or Lebesgue space $L^p(\Omega)$ has been studied in Boman [6] as well as Crouzeix and Thomée [13], in the space of functions of bounded variation in Cockburn [12], and in the fractional Sobolev space $H^s(\Omega)$ with $s \in (0, 1]$ in Steinbach [42] (the case $s = 1$ has also been studied by several other authors). Assumption P is also satisfied when $H = H^{-1}(\Omega)$ with $\Omega = (a, b) \subset \mathbb{R}$, $V = L^p(\Omega)$ and $V_m$ consists of piecewise constant functions, see Emmrich and Šiška [19]. Finally, if $V = V_A$, one may also use a Galerkin basis that consists of eigenfunctions of the operator $A$.

**Assumption IC** (Initial conditions)**.** *Let $u_0 \in V_B$ and $v_0 \in H$. Let there be sequences $\{u_\ell^0\}_{\ell \in \mathbb{N}}$ and $\{v_\ell^0\}_{\ell \in \mathbb{N}}$ such that $u_\ell^0$ and $v_\ell^0$ lie in $V_{m_\ell}$ for all $\ell \in \mathbb{N}$ and such that $u_\ell^0 \to u_0$ in $V_B$ and $v_\ell^0 \to v_0$ in $H$ as $\ell \to \infty$. Let there be $c > 0$ such that $\tau_\ell \|v_\ell^0\|_{V_A}^2 < c$ for all $\ell \in \mathbb{N}$.*

Note that the last condition, which later simplifies the application of the Lions–Aubin lemma, can always be fulfilled since $V_A$ is dense in $H$.

For the right-hand side $f \in L^2(0, T; V_A^*)$, we use the approximation

$$f^n := \frac{1}{\tau} \int_{t_{n-1}}^{t_n} f(t) dt, \quad n = 1, \ldots, N.$$

Given $\tau_\ell$, an approximation to the right-hand side $\{f^n\}_{n=1}^{N_\ell}$ and the solution $\{u^n\}_{n=0}^{N_\ell} \subset V_{m_\ell}$, $\{v^n\}_{n=0}^{N_\ell} \subset V_{m_\ell}$ to (2.7), we define the piecewise constant abstract functions

$$f_\ell(t) := f^n, \quad u_\ell(t) := u^n, \quad v_\ell(t) := v^n \ \text{ for } \ t \in (t_{n-1}, t_n],\ n = 1, \ldots, N_\ell,$$

as well as the piecewise linear and continuous abstract functions

$$\begin{aligned} \hat{u}_\ell(t) &:= u^{n-1} + \frac{t - t_{n-1}}{\tau_\ell}\left(u^n - u^{n-1}\right), \\ \hat{v}_\ell(t) &:= v^{n-1} + \frac{t - t_{n-1}}{\tau_\ell}\left(v^n - v^{n-1}\right) \qquad\qquad \text{for } \ t \in (t_{n-1}, t_n],\ n = 1, \ldots, N_\ell. \end{aligned}$$

Here, as well as in the remainder of the paper, we often write $t_n$, $u^n$, $v^n$ and $f^n$ instead of $t_{n,\ell}$, $u_\ell^n$, $v_\ell^n$ and $f_\ell^n$. Note that $\hat{u}_\ell' = v_\ell$ and that, as one can easily show, $f_\ell \to f$ in $L^2(0,T;V_A^*)$ as $\ell \to \infty$.

We can now rewrite the discrete problem (2.7) as

$$\langle \hat{v}_\ell'(t), \varphi\rangle + \langle A v_\ell(t), \varphi\rangle + \langle B u_\ell(t), \varphi\rangle = \langle f_\ell(t), \varphi\rangle \quad \forall \varphi \in V_{m_\ell}, \tag{4.1}$$

which holds for almost all $t \in (0,T)$ as well as in the weak sense on $(0,T)$.

**Definition 4.1** (Solution)**.** *Let $u_0 \in V_B$, $v_0 \in H$ and $f \in L^2(0,T;V_A^*)$. A function $u \in \mathscr{C}_w([0,T];V_B)$ with $u' \in \mathscr{C}_w([0,T];H) \cap L^2(0,T;V_A)$ and $u'' \in L^2(0,T;V^*)$ is said to be a solution to* (1.1) *provided that the first equality in* (1.1) *is satisfied in $L^2(0,T;V_A^*)$ and that $u(0) = u_0$ in $V_B$ as well as $u'(0) = v_0$ in $H$.*

Note that, in general, $u''$ only takes values in $V^*$ but $u'' + Bu = f - Au$ takes values in $V_A^*$. Let us remark that in the nonconvex case we are only able to prove existence of a solution under the additional assumption $u_0 \in V_A$, which then implies $u \in \mathscr{AC}([0,T];V_A)$.

Now we may state the main result of this paper.

**Theorem 4.2** (Existence for continuous problem and convergence of full discretizations)**.** *Let Assumptions* A, B, AB, IC *and* P *hold, let $V_A$ be compactly embedded in $H$ and let $f \in L^2(0,T;V_A)$. If $\phi_B : V_B \to \mathbb{R}$ is not convex (i.e., $\lambda > 0$ in Assumption* AB*) assume, in addition, that $u_0 \in V_A$. Then there is a solution $u$ to* (1.1) *according to Definition 4.1.*

*Moreover, there is a subsequence of the sequence of approximate solutions, denoted by $\ell'$, such that $u_{\ell'} - u_{\ell'}^0$ and $\hat{u}_{\ell'} - u_{\ell'}^0$ both converge strongly in $L^2(0,T;V_A)$ and weakly* in $L^\infty(0,T;V)$ towards $u - u_0$, $v_{\ell'} = \hat{u}_{\ell'}'$ and $\hat{v}_{\ell'}$ both converge strongly in $L^2(0,T;H)$, weakly* in $L^\infty(0,T;H)$ and weakly in $L^2(0,T;V_A)$ towards $u'$, and $\hat{v}_{\ell'}'$ converges weakly in $L^2(0,T;V^*)$ towards $u''$ as $\ell' \to \infty$ provided that $u_\ell^0 \to u_0$ in $V_A$ as $\ell \to \infty$ and $\tau_\ell \le \mu_A/(2\lambda c_A)$ if $\phi_B : V_B \to \mathbb{R}$ is not convex.*

The proof of the above theorem will be prepared by several auxiliary results and finally finished at the end of this section. Here we give a short outline of our method. The a priori estimates (Theorem 3.4) will allow us to use compactness arguments to extract a subsequence of approximate solutions converging weakly towards $u$. With this at hand, the main difficulty will be in the passage to the limit in the nonlinear term. Initially, we will only be able to conclude that the nonlinear term converges weakly* to some $b \in L^\infty(0,T;V_B^*)$. In order to identify $b$ with $Bu$, we will first use the Lions–Aubin lemma to obtain a subsequence of approximations that converges strongly in $L^2(0,T;H)$. From this point onwards, we also need to assume that $u_0 \in V_A$ in order to apply the generalized Andrews–Ball-type condition in the form of the monotonicity of $(B + \lambda A) : V \to V^*$ for some $\lambda > 0$ if the potential $\phi_B : V_B \to \mathbb{R}$ is not convex. This way, we obtain a subsequence of approximate solutions converging strongly in $L^2(0, T; V_A)$ towards $u$. Finally, we will be able to identify the limit of the nonlinear term with $Bu$ by using the monotonicity of $(B + \lambda A) : V \to V^*$ in a Minty-type monotonicity argument together with an appropriate integration-by-parts formula.

*4.2. Convergent subsequence from a priori estimates and the limit equation*

We will use the a priori estimates for the discrete problem together with compactness arguments to obtain a weakly convergent subsequence of interpolations of solutions to the discrete problem.

**Lemma 4.3.** *Let Assumptions* A, B, AB *and* IC *hold and let* $\tau_\ell \leq \mu_A/(2\lambda c_A)$ *if* $\lambda > 0$. *Then there exists a subsequence, denoted by* $\ell'$, *and some*

$$\begin{aligned} &u \in \mathscr{C}_w([0, T]; V_B) \cap \mathscr{AC}([0, T]; H) \text{ with } u(0) = u_0 \in V_B \\ &\text{and } u - u_0 \in \mathscr{AC}([0, T]; V_A),\ u' \in L^2(0, T; V_A) \cap L^\infty(0, T; H) \end{aligned}$$

*such that, as* $\ell' \to \infty$,

$$\begin{aligned} &u_{\ell'}, \hat{u}_{\ell'} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0, T; V_B),\ u_{\ell'} - u^0_{\ell'}, \hat{u}_{\ell'} - u^0_{\ell'} \overset{*}{\rightharpoonup} u - u_0 \text{ in } L^\infty(0, T; V_A), \\ &\hat{u}_\ell - u_\ell \to 0 \text{ in } L^2(0, T; V_A), \\ &v_{\ell'}, \hat{v}_{\ell'} \overset{*}{\rightharpoonup} u' \text{ in } L^\infty(0, T; H),\ v_{\ell'}, \hat{v}_{\ell'} \rightharpoonup u' \text{ in } L^2(0, T; V_A), \\ &\hat{v}_\ell - v_\ell \to 0 \text{ in } L^2(0, T; H). \end{aligned}$$

*If, in addition, Assumption* P *holds then*

$$u' \in \mathscr{C}_w([0, T]; H) \cap \mathscr{AC}([0, T]; V^*) \text{ with } u'(0) = v_0 \in H \text{ and } u'' \in L^2(0, T; V^*)$$

*and, as* $\ell' \to \infty$,

$$\begin{aligned} &\hat{v}'_{\ell'} \rightharpoonup u'' \text{ in } L^2(0, T; V^*), \\ &\hat{v}_{\ell'}(T) = v_{\ell'}(T) = v^{N_{\ell'}}_{\ell'} \rightharpoonup u'(T),\ \hat{v}_{\ell'}(t) \rightharpoonup u'(t)\, (t \in [0, T]) \text{ in } H. \end{aligned}$$

*If, moreover,* $V_A$ *is compactly embedded in* $H$ *then, as* $\ell' \to \infty$,

$$\begin{aligned} &v_{\ell'}, \hat{v}_{\ell'} \to u' \text{ in } L^2(0, T; H), \\ &\hat{u}_{\ell'}(T) = u_{\ell'}(T) = u^{N_{\ell'}}_{\ell'} \to u(T),\ \hat{u}_{\ell'} \to u \text{ in } \mathscr{C}([0, T]; H). \end{aligned}$$

*Proof.* We begin by observing that

$$\tau_\ell \sum_{n=1}^{N_\ell} \|f^n\|^2_{V_A^*} \leq \|f\|^2_{L^2(0,T;V_A)}.$$

Furthermore, since $\{v^0_\ell\}_{\ell \in \mathbb{N}}$ is bounded in $H$ and $\{u^0_\ell\}_{\ell \in \mathbb{N}}$ is bounded in $V_B$ and recalling that $\phi_B : V_B \to \mathbb{R}$ is bounded, the right-hand sides of both the inequalities in Theorem 3.4 are bounded by a constant independent of $\ell$.

Therefore, we have $\phi_B(u_\ell(t)) \leq c$ with $c$ independent of $\ell$ and $t$. The weak coercivity of $\phi_B : V_B \to \mathbb{R}$ then implies that $\{u_\ell\}_{\ell \in \mathbb{N}}$, and thus also $\{\hat{u}_\ell\}_{\ell \in \mathbb{N}}$, is bounded in $L^\infty(0, T; V_B)$. As $V_B$ is separable and reflexive, we have $V_B^*$ separable and reflexive (see, e.g., Brézis [7, Corollary 3.27]). Due to Diestel and Uhl [17, Chapter IV, Section 1, Theorem 1 with Chapter III, Section 3, Theorem 1], $L^\infty(0,T;V_B)$ is the dual of the separable space $L^1(0,T;V_B^*)$. Hence, there are a subsequence, denoted by $\ell'$, and elements $u, \hat{u} \in L^\infty(0,T;V_B)$ such that $u_{\ell'} \overset{*}{\rightharpoonup} u$ and $\hat{u}_{\ell'} \overset{*}{\rightharpoonup} \hat{u}$ in $L^\infty(0,T;V_B)$ as $\ell' \to \infty$ (see, e.g., Brézis [7, Corollary 3.30]).

In view of the second inequality in Theorem 3.4, both $\{u_\ell - u_\ell^0\}_{\ell\in\mathbb{N}}$ and $\{\hat{u}_\ell - u_\ell^0\}_{\ell\in\mathbb{N}}$ are bounded in $L^\infty(0,T;V_A)$. This implies that there is a subsequence (of the subsequence, still denoted by $\ell'$) such that $u_{\ell'} - u_{\ell'}^0$ and $\hat{u}_{\ell'} - u_{\ell'}^0$ are weakly* convergent in $L^\infty(0,T;V_A)$. Since $u_\ell^0 \to u_0$ in $V_B$ by assumption, the limits can only be $u - u_0$ and $\hat{u} - u_0$, respectively.

A simple calculation reveals that

$$\|\hat{u}_\ell - u_\ell\|_{L^2(0,T;V_A)}^2 = \frac{\tau_\ell}{3}\sum_{n=1}^{N_\ell}\tau_\ell^2\left\|\frac{u^n - u^{n-1}}{\tau_\ell}\right\|_{V_A}^2 = \frac{\tau_\ell^2}{3}\sum_{n=1}^{N_\ell}\tau_\ell\|v^n\|_{V_A}^2 \to 0 \ \text{ as } \ \ell \to \infty,$$

because of the a priori estimate (3.1a). This implies $u = \hat{u}$.

From the a priori estimate (3.1a), we see that $\{v_\ell\}_{\ell\in\mathbb{N}}$ and $\{\hat{v}_\ell\}_{\ell\in\mathbb{N}}$ are bounded in $L^\infty(0,T;H)$. Hence, as before, there are a subsequence of the subsequence, still denoted by $\ell'$, and elements $v$, $\hat{v}$ in $L^\infty(0,T;H)$ such that $v_{\ell'} \overset{*}{\rightharpoonup} v$ and $\hat{v}_{\ell'} \overset{*}{\rightharpoonup} \hat{v}$ in $L^\infty(0,T;H)$ as $\ell' \to \infty$.

Furthermore, the sequence $\{v_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^2(0,T;V_A)$. Next, we notice that

$$\|\hat{v}_\ell\|_{L^2(0,T;V_A)}^2 \le c\sum_{n=1}^{N_\ell}\tau_\ell\|v^n\|_{V_A}^2 + c\tau_\ell\|v_\ell^0\|_{V_A}^2.$$

This and the assumption that $\tau_\ell\|v_\ell^0\|_{V_A}^2 \le c$ shows that also $\{\hat{v}_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^2(0,T;V_A)$. As $V_A$ is reflexive, $L^2(0,T;V_A)$ is also reflexive and so (by, e.g., Brézis [7, Theorem 3.18]) there is a subsequence of the subsequence, still denoted by $\ell'$, such that $v_{\ell'} \rightharpoonup v$ and $\hat{v}_{\ell'} \rightharpoonup \hat{v}$ in $L^2(0,T;V_A)$ as $\ell' \to \infty$.

Moreover, we observe that

$$\|\hat{v}_\ell - v_\ell\|_{L^2(0,T;H)}^2 = \frac{\tau_\ell}{3}\sum_{n=1}^{N_\ell}|v^n - v^{n-1}|^2 \to 0 \text{ as } \ell \to \infty \tag{4.2}$$

because of the a priori estimate (3.1a). Thus $v = \hat{v}$.

Since $\hat{u}_\ell' = v_\ell$ and $\hat{u}_{\ell'} \overset{*}{\rightharpoonup} u$ as well as $v_{\ell'} \overset{*}{\rightharpoonup} v$ in $L^\infty(0,T;H)$ as $\ell' \to \infty$, it follows, using the definition of the weak derivative of a function with values in $H$, that $v = u'$. Next, we observe that, since $u \in L^\infty(0,T;V_B)$ and $v = u' \in L^\infty(0,T;H)$, we get $u \in \mathscr{AC}([0,T],H)$. Due to Lions and Magenes [31, Chapitre 3, Lemme 8.1], we thus have $u \in \mathscr{C}_w([0,T];V_B)$. Furthermore, $u - u_0 \in L^\infty(0,T;V_A)$ and $v = u' = (u-u_0)' \in L^2(0,T;V_A)$ implies $u - u_0 \in H^1(0,T;V_A) \subseteq \mathscr{AC}([0,T],V_A)$.

Since $H^1(0,T;V_A) \hookrightarrow \mathscr{C}([0,T];V_A)$, we can consider the trace operator $\Gamma_0 : H^1(0,T;V_A) \to V_A$ with $\Gamma_0 w = w(0)$, which is linear and bounded and thus weakly-weakly continuous (see, e.g., Brézis [7, Theorem 3.10]). As $\hat{u}_{\ell'} - u_{\ell'}^0 \rightharpoonup u - u_0$ and $(\hat{u}_{\ell'} - u_{\ell'}^0)' = v_{\ell'} \rightharpoonup u' = (u-u_0)'$ in $L^2(0,T;V_A)$ as $\ell' \to \infty$, we thus find $(\hat{u}_{\ell'} - u_{\ell'}^0)(0) \rightharpoonup (u-u_0)(0)$ in $V_A$ as $\ell' \to \infty$ but $(\hat{u}_{\ell'} - u_{\ell'}^0)(0) \equiv 0$. This shows that $(u-u_0)(0) = 0$ in $V_A$. On the other hand, we already know that $u \in \mathscr{C}_w([0,T];V_B)$ such that $u(t) \rightharpoonup u(0)$ in $V_B$ as $t \to 0$ and hence $u(0) = u_0$ in $V_B$.

If Assumption P holds then, in view of Theorem 3.5, since $\{u_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V_B)$ and thus $\{Bu_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^\infty(0,T;V_B^*)$, and since

$$\|\hat{v}_\ell'\|^2_{L^2(0,T;V^*)} = \tau_\ell \sum_{n=1}^{N_\ell} \left\| \frac{v^n - v^{n-1}}{\tau_\ell} \right\|^2_{V^*} \leq c,$$

the sequence $\{\hat{v}_\ell'\}_{\ell\in\mathbb{N}}$ is bounded in $L^2(0,T;V^*)$. This shows that, again for a subsequence denoted by $\ell'$, $\hat{v}_{\ell'}' \rightharpoonup v' = u''$ in $L^2(0,T;V^*)$ as $\ell' \to \infty$. Since then $u'' \in L^2(0,T;V^*)$ and $u' \in L^\infty(0,T;H)$, we find $u' \in \mathscr{AC}([0,T];V^*)$ and thus $u' \in \mathscr{C}_w([0,T];H)$.

We also see that $\hat{v}_{\ell'} \rightharpoonup u'$ in $H^1(0,T;V^*) \hookrightarrow \mathscr{C}([0,T];V^*)$ and thus, again by employing the weak-weak continuity of the corresponding trace operator, $\hat{v}_{\ell'}(0) \rightharpoonup u'(0)$ in $V^*$ as $\ell' \to \infty$. However, we have $\hat{v}_\ell(0) = v_\ell^0 \to v_0$ in $H$ as $\ell \to \infty$ by assumption and $u' \in \mathscr{C}_w([0,T];H)$, which shows that $u'(0) = v_0$ in $H$. An analogous argumentation shows that $\hat{v}_{\ell'}(T) = v_{\ell'}(T) = v_{\ell'}^{N_{\ell'}} \rightharpoonup u'(T)$ as well as $\hat{v}_{\ell'}(t) \rightharpoonup u'(t)$ in $H$ for all $t \in [0,T]$ as $\ell' \to \infty$.

We will now use the assumption that $V_A$ is compactly embedded in $H$. Consider the Banach space

$$\mathscr{Z} := \left\{ w \in L^2(0,T;V_A) : w' \in L^2(0,T;V^*) \right\}, \quad \|w\|_{\mathscr{Z}} := \|w\|_{L^2(0,T;V_A)} + \|w'\|_{L^2(0,T;V^*)}.$$

The generalized Lions–Aubin lemma (see Roubíček [41, Lemma 7.7]) implies that $\mathscr{Z}$ is compactly embedded in $L^2(0,T;H)$. We have shown that $\{\hat{v}_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $\mathscr{Z}$. Hence there is a subsequence of the subsequence, still denoted by $\ell'$, such that $\hat{v}_{\ell'} \to v = u'$ in $L^2(0,T;H)$ as $\ell' \to \infty$. In view of estimate (4.2), we also obtain $v_{\ell'} \to v = u'$ in $L^2(0,T;H)$ as $\ell' \to \infty$.

Furthermore, we have for all $t \in [0,T]$

$$|\hat{u}_{\ell'}(t) - u(t)| \leq |u_{\ell'}^0 - u_0| + \int_0^t |\hat{u}_{\ell'}'(s) - u'(s)| ds \leq |u_{\ell'}^0 - u_0| + \|\hat{u}_{\ell'}' - u'\|_{L^1(0,T;H)}.$$

Since $\hat{u}_{\ell'}' = v_{\ell'}$ and since $v_{\ell'} \to u'$ in $L^2(0,T;H)$ as $\ell' \to \infty$, this finishes the proof (recalling that $\hat{u}_{\ell'}(T) = u_{\ell'}(T) = u^{N_{\ell'}}$). ☐

We will now pass to the limit in (4.1).

**Lemma 4.4.** *Under the assumptions of Theorem 4.2 there are a subsequence, denoted by $\ell'$, and some $b \in L^\infty(0,T;V_B^*)$ such that $Bu_{\ell'} \overset{*}{\rightharpoonup} b$ in $L^\infty(0,T;V_B^*)$ as $\ell' \to \infty$, and the limit $u$ obtained in Lemma 4.3 satisfies*

$$u'' + Au' + b = f \quad \text{in } L^2(0,T;V_A^*). \tag{4.3}$$

*Proof.* Let $\{u^n\}_{n=0}^{N_\ell} \subset V_{m_\ell}$, $\{v^n\}_{n=0}^{N_\ell} \subset V_{m_\ell}$ denote the solution to (2.7). Equation (4.1) then implies

$$-\int_0^T (\hat{v}_\ell(t),\varphi)\psi'(t)dt + \int_0^T \langle Av_\ell(t),\varphi\rangle\psi(t)dt + \int_0^T \langle Bu_\ell(t),\varphi\rangle\psi(t)dt = \int_0^T \langle f_\ell(t),\varphi\rangle\psi(t)dt \tag{4.4}$$

for all $\varphi \in V_k$, with $k \leq m_\ell$ fixed, and all $\psi \in \mathscr{C}_c^\infty(0,T)$. The lemma will be proved by taking the limit in (4.4) along a subsequence of $\ell'$ while keeping $k$ fixed.

First we observe that, due to the a priori estimates in Theorem 3.4 and Assumption B,

$$\|Bu_\ell\|_{L^\infty(0,T;V_B^*)} = \max_{n=1,\dots,N_\ell} \|Bu_\ell^n\|_{V_B^*}$$

is bounded uniformly in $\ell$. Indeed, the weak coercivity of the potential $\phi_B : V_B \to \mathbb{R}$ and (3.1a) imply the boundedness of the set $\{\|u_\ell^n\|_{V_B} : n = 1, \ldots, N_\ell;\ \ell \in \mathbb{N}\}$. Moreover, $B : V_B \to V_B^*$ is a bounded operator.

As $V_B$ is separable, the Bochner–Lebesgue space $L^1(0,T;V_B)$ is separable and so $L^\infty(0,T;V_B^*)$ is the dual of a separable Banach space. Then due to, e.g., Brézis [7, Corollary 3.30] there are a subsequence of the subsequence from the previous lemma, still denoted by $\ell'$, and an element $b \in L^\infty(0,T;V_B^*)$ such that $Bu_{\ell'} \overset{*}{\rightharpoonup} b$ in $L^\infty(0,T;V_B^*)$ as $\ell' \to \infty$.

Because $A : L^2(0,T;V_A) \to L^2(0,T;V_A^*)$ is weakly-weakly continuous, we have $Av_{\ell'} \rightharpoonup Au'$ in $L^2(0,T;V_A^*)$ since $v_{\ell'} \rightharpoonup u'$ in $L^2(0,T;V_A)$ as $\ell' \to \infty$. Moreover, we have $\hat{v}_{\ell'} \overset{*}{\rightharpoonup} u'$ in $L^\infty(0,T;H)$ as well as $f_\ell \to f$ in $L^2(0,T;V_A^*)$ as $\ell' \to \infty$.

Hence, letting $\ell' \to \infty$ in (4.4) while keeping $k$ fixed, we obtain

$$-\int_0^T (u'(t),\varphi)\psi'(t)dt + \int_0^T \langle Au'(t),\varphi\rangle\psi(t)dt + \int_0^T \langle b(t),\varphi\rangle\psi(t)dt = \int_0^T \langle f(t),\varphi\rangle\psi(t)dt \tag{4.5}$$

for all $\varphi \in V_k$ and all $\psi \in \mathscr{C}_c^\infty(0,T)$. Now we use the limited completeness of the Galerkin scheme $\{V_k\}_{k\in\mathbb{N}}$ in $V$ and let $k \to \infty$ to obtain the above equality, but this time for all $\varphi \in V$ and all $\psi \in \mathscr{C}_c^\infty(0,T)$.

Equation (4.5) then shows that $f - Au' - b \in L^2(0,T;V_A^*) + L^\infty(0,T;V_B^*) \subseteq L^2(0,T;V^*)$ is the weak derivative of $u' \in L^2(0,T;V_A) \subseteq L^2(0,T;V^*)$ (see, e.g., Temam [43, Lemma 1.1 on p. 250]). We, therefore, obtain (4.3) since the set of functions $t \mapsto \varphi\psi(t)$ with $\varphi \in V$ and $\psi \in \mathscr{C}_c^\infty(0,T)$ is dense in $L^2(0,T;V)$. Equation (4.3) indeed holds in $L^2(0,T;V_A^*)$ since $u'' + b = f - Au \in L^2(0,T;V_A^*)$. □

*4.3. Discrete integration by parts*

In the sequel, we will need the following crucial fact, which is based on a discrete integration-by-parts formula reflecting the stability of the time discretization scheme.

**Lemma 4.5.** *Let the assumptions of Theorem 4.2 hold. Then for all $t \in [0,T]$*

$$\int_0^t (\hat{v}_{\ell'}'(s), u_{\ell'}(s) - u_{\ell'}^0)ds \to (u'(t), u(t) - u_0) - \int_0^t |u'(s)|^2 ds = \int_0^t \langle u''(s), u(s) - u_0\rangle ds \quad \textit{as } \ell' \to \infty.$$

*Proof.* In what follows, we only write $\ell$ instead of $\ell'$. We observe that

$$\int_0^t (\hat{v}_\ell'(s), u_\ell(s) - u_\ell^0)ds = \int_0^t (\hat{v}_\ell'(s), \hat{u}_\ell(s) - u_\ell^0)ds + \int_0^t (\hat{v}_\ell'(s), u_\ell(s) - \hat{u}_\ell(s))ds\,. \tag{4.6}$$

For the first term on the right-hand side, we can carry out integration by parts and obtain with $(\hat{u}_\ell - u_\ell^0)' = v_\ell$ and $\hat{u}_\ell(0) = u_\ell^0$

$$\int_0^t (\hat{v}_\ell'(s), \hat{u}_\ell(s) - u_\ell^0)ds = (\hat{v}_\ell(t), \hat{u}_\ell(t) - u_\ell^0) - \int_0^t (\hat{v}_\ell(s), v_\ell(s))ds\,.$$

In view of Lemma 4.3 and Assumption IC, we thus immediately get

$$\int_0^t (\hat{v}_\ell'(s), \hat{u}_\ell(s) - u_\ell^0)ds \to (u'(t), u(t) - u_0) - \int_0^t |u'(s)|^2 ds \ \text{ as } \ \ell \to \infty.$$

We now consider the second term on the right-hand side of (4.6). Note that $\{\hat{v}'_\ell\}_{\ell\in\mathbb{N}}$ is bounded in $L^2(0,T;V^*)$ but $u_\ell - \hat{u}_\ell$ strongly converges towards zero only in $L^2(0,T;V_A)$ (and only weakly in $L^\infty(0,T;V_B)$). Therefore, we cannot pass to the limit immediately. However, we observe the following.

Let $t \in (t_{n-1}, t_n]$ for some $n \in \{1, \ldots, N_\ell\}$. We then find (recalling that $v^j = (u^j - u^{j-1})/\tau_\ell$)

$$\begin{aligned}
\left|\int_0^t (\hat{v}'_\ell(s), u_\ell(s) - \hat{u}_\ell(s))ds\right| &\le \int_0^{t_n} |\hat{v}'_\ell(s)|\,|u_\ell(s) - \hat{u}_\ell(s))|ds \\
&= \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \left|\frac{v^j - v^{j-1}}{\tau_\ell}\right| \left|\frac{u^j - u^{j-1}}{\tau_\ell}(t_j - s)\right| ds \\
&= \frac{\tau_\ell}{2} \sum_{j=1}^n |v^j - v^{j-1}|\,|v^j| \\
&\le \frac{\tau_\ell^{1/2} T^{1/2}}{2} \left(\sum_{j=1}^{N_\ell} |v^j - v^{j-1}|^2\right)^{1/2} \max_{j=1,\ldots,N_\ell} |v^j|\,.
\end{aligned}$$

The right-hand side of the foregoing estimate converges, in view of Theorem 3.4, towards zero as $\ell \to \infty$.

Lemma 5.1 in the appendix finally proves the assertion since $u - u_0 \in L^2(0,T;V)$ with $u' = (u - u_0)' \in L^2(0,T;V_A) \cap L^\infty(0,T;H)$ and $u'' = (u - u_0)'' \in L^2(0,T;V^*)$. □

*4.4. Strong convergence and identification of the nonlinear term*

All that remains to be done in order to prove Theorem 4.2 is to identify $b$ with $Bu$. The main idea in identifying $b$ with $Bu$ is to test equation (4.1) with $u_\ell$, to use then the generalized Andrews–Ball condition (Assumption AB) and to apply a variant of Minty's monotonicity trick. In order to do so, we first have to prove strong convergence of the approximate solutions in $L^2(0,T;V_A)$. This is provided by the following lemma. In the situation of the example (1.2), the result was shown in Prohl [37].

**Lemma 4.6.** *Let the assumptions of Theorem 4.2 hold. Then $u_{\ell'} - u^0_{\ell'} \to u - u_0$ in $L^2(0,T;V_A)$ as $\ell' \to \infty$.*

Note that, so far, $u$ is a function taking values in $V_B$ but $u - u_0$ takes as well values in $V_A$. We emphasize that, in the nonconvex case, we finally have to assume $u^0_\ell \to u_0$ in $V$ as $\ell \to \infty$ and immediately find that $u$ takes values in $V_A$. The assertion of the foregoing lemma then implies $u_{\ell'} \to u$ in $L^2(0,T;V_A)$ as $\ell' \to \infty$, which is crucial for the existence proof in the nonconvex case.

Orthogonal projections $Q_{m_\ell} : V_A \to V_{m_\ell}$ will be used in the proof of the above lemma. As $A : V_A \to V_A^*$ is a linear, bounded, strongly positive and symmetric operator, the space $V_A$ is a Hilbert space with an inner product that is equivalent to $\langle A\cdot, \cdot\rangle$. Hence, for each $V_{m_\ell}$, the orthogonal projection $Q_{m_\ell} : V_A \to V_{m_\ell}$ with $Q_{m_\ell} w$ defined by

$$\langle A Q_{m_\ell} w, \varphi\rangle = \langle A w, \varphi\rangle \quad \forall \varphi \in V_{m_\ell}$$

exists. We point out that its operator norm as an operator in $V_A$ equals one if we use the operator norm induced by $\|\cdot\|_A = \langle A\cdot, \cdot\rangle^{1/2}$. Recall that this norm is equivalent to $\|\cdot\|_{V_A}$. Furthermore, the orthogonal projection $Q_{m_\ell} : V_A \to V_{m_\ell}$ has the following properties:

1. It gives the best approximation of $w \in V_A$ in the space $V_{m_\ell}$ in the sense that
$$\|Q_{m_\ell} w - w\|_A \leq \inf_{z \in V_{m_\ell}} \|z - w\|_A \quad \forall w \in V_A.$$
2. Since $\{V_m\}_{m\in\mathbb{N}}$ is a Galerkin scheme for $V$ and since $V$ is continuously and densely embedded in $V_A$, it can be shown that $Q_{m_\ell} w \to w$ in $V_A$ as $\ell \to \infty$. Let $w \in L^2(0,T;V_A)$. It can then be shown that $Q_{m_\ell} w \to w$ in $L^2(0,T;V_A)$ as $\ell \to \infty$, where $Q_{m_\ell} w : [0,T] \to V_A$ is defined by $(Q_{m_\ell} w)(t) := Q_{m_\ell} w(t)$.
3. Let $w \in H^1(0,T;V_A)$. We then find $Q_{m_\ell} w \in H^1(0,T;V_A)$ and $(Q_{m_\ell} w)' = Q_{m_\ell} w'$ in the weak sense.

*Proof of Lemma 4.6.* We only write $\ell$ instead of $\ell'$. Let $z_\ell := \hat{u}_\ell - u_\ell^0 - Q_{m_\ell}(u - u_0)$. We then obtain
$$\|u_\ell - u_\ell^0 - (u - u_0)\|_{L^2(0,T;V_A)} \leq \|u_\ell - \hat{u}_\ell\|_{L^2(0,T;V_A)} + \|z_\ell\|_{L^2(0,T;V_A)} + \|Q_{m_\ell}(u-u_0) - (u-u_0)\|_{L^2(0,T;V_A)}. \quad (4.7)$$
Since the first and last term on the right-hand side of the foregoing estimate goes to zero as $\ell \to \infty$ (see Lemma 4.3 for the first and employ the properties of $Q_{m_\ell}$ for the last term), we focus on the term with $z_\ell$.

As $z_\ell \in L^2(0,T;V_A)$ with $z_\ell' = v_\ell - Q_{m_\ell} u' \in L^2(0,T;V_A)$, we find by employing the symmetry of $A$, the definition of $Q_{m_\ell}$ and (4.1)
$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z_\ell(t)\|_A^2 &= \langle A(v_\ell(t) - Q_{m_\ell} u'(t)), z_\ell(t)\rangle \\
&= \langle A v_\ell(t), \hat{u}_\ell(t) - u_\ell^0\rangle - \langle A v_\ell(t), u(t) - u_0\rangle - \langle A u'(t), z_\ell(t)\rangle \\
&= \langle A v_\ell(t), u_\ell(t) - u_\ell^0\rangle + \langle A v_\ell(t), \hat{u}_\ell(t) - u_\ell(t)\rangle - \langle A v_\ell(t), u(t) - u_0\rangle - \langle A u'(t), z_\ell(t)\rangle \\
&= -\langle B u_\ell(t), u_\ell(t)\rangle + \langle B u_\ell(t), u_\ell^0\rangle - \langle \hat{v}_\ell'(t), u_\ell(t) - u_\ell^0\rangle + \langle f_\ell(t), u_\ell(t) - u_\ell^0\rangle \\
&\quad + \langle A v_\ell(t), \hat{u}_\ell(t) - u_\ell(t)\rangle - \langle A v_\ell(t), u(t) - u_0\rangle - \langle A u'(t), z_\ell(t)\rangle \\
&= -\langle B u_\ell(t) - B u(t), u_\ell(t) - u(t)\rangle + I_\ell(t),
\end{aligned} \quad (4.8)$$
where
$$\begin{aligned}
I_\ell(t) := &-\langle B u_\ell(t), u(t)\rangle - \langle B u(t), u_\ell(t) - u(t)\rangle + \langle B u_\ell(t), u_\ell^0\rangle - \langle \hat{v}_\ell'(t), u_\ell(t) - u_\ell^0\rangle \\
&+ \langle f_\ell(t), u_\ell(t) - u_\ell^0\rangle + \langle A v_\ell(t), \hat{u}_\ell(t) - u_\ell(t)\rangle - \langle A v_\ell(t), u(t) - u_0\rangle - \langle A u'(t), z_\ell(t)\rangle.
\end{aligned}$$
Note that
$$\begin{aligned}
\left|\int_0^t I_\ell(s)ds\right| \leq \int_0^T |I_\ell(s)|ds \leq\ & \|B u_\ell\|_{L^2(0,T;V_B^*)}\|u\|_{L^2(0,T;V_B)} + \|Bu\|_{L^2(0,T;V_B^*)}\|u_\ell - u\|_{L^2(0,T;V_B)} \\
&+ \|B u_\ell\|_{L^1(0,T;V_B^*)}\|u_\ell^0\|_{V_B} + \|\hat{v}_\ell'\|_{L^2(0,T;V^*)}\|u_\ell - u_\ell^0\|_{L^2(0,T;V)} \\
&+ \|f_\ell\|_{L^2(0,T;V_A^*)}\|u_\ell - u_\ell^0\|_{L^2(0,T;V_A)} + \|A v_\ell\|_{L^2(0,T;V_A^*)}\|\hat{u}_\ell - u_\ell\|_{L^2(0,T;V_A)} \\
&+ \|A v_\ell\|_{L^2(0,T;V_A^*)}\|u - u_0\|_{L^2(0,T;V_A)} + \|A u'\|_{L^2(0,T;V_A^*)}\|z_\ell\|_{L^2(0,T;V_A)},
\end{aligned}$$
where the right-hand side is uniformly bounded due to the a priori estimates in Theorem 3.4, 3.5 and due to the estimate
$$\|z_\ell(t)\|_A \leq \|\hat{u}_\ell(t) - u_\ell^0\|_A + \|Q_{m_\ell}(u(t) - u_0)\|_A \leq c\,\|\hat{u}_\ell - u_\ell^0\|_{L^\infty(0,T;V_A)} + \|u(t) - u_0\|_A.$$

Let $\phi_B : V_B \to \mathbb{R}$ be convex such that $B : V_B \to V_B^*$ is monotone (i.e., $\lambda = 0$ in Assumption AB). Then (4.8) implies for all $t \in [0,T]$ (because of $z_\ell(0) = 0$)

$$\frac{1}{2}\|z_\ell(t)\|_A^2 \leq \int_0^t I_\ell(s)ds. \tag{4.9}$$

In view of Lemma 4.3, 4.4 and 4.5 we already know that, as $\ell \to \infty$,

$$\begin{aligned}
&Bu_\ell \overset{*}{\rightharpoonup} b \text{ in } L^\infty(0,T;V_B^*),\ u_\ell \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T;V_B),\ u_\ell^0 \to u_0 \text{ in } V_B,\\
&\int_0^t \langle \hat{v}_\ell'(s), u_\ell(s) - u_\ell^0\rangle ds \to \int_0^t \langle u''(s), u(s) - u_0\rangle ds,\\
&f_\ell \to f \text{ in } L^2(0,T;V_A^*),\ u_\ell - u_\ell^0 \rightharpoonup u - u_0 \text{ in } L^2(0,T;V_A),\ Av_\ell \rightharpoonup Au' \text{ in } L^2(0,T;V_A^*),\\
&\hat{u}_\ell - u_\ell \to 0 \text{ in } L^2(0,T;V_A),\ z_\ell \rightharpoonup 0 \text{ in } L^2(0,T;V_A).
\end{aligned}$$

All this implies $\int_0^t I_\ell(s)ds \to 0$ as $\ell \to \infty$. This and the uniform boundedness of $\max_{t\in[0,T]}\left|\int_0^t I_\ell(s)ds\right|$ allow us to apply Lebesgue's theorem on dominated convergence, which provides $\int_0^T \int_0^t I_\ell(s)dsdt \to 0$ and thus, because of (4.9), the strong convergence $z_\ell \to 0$ in $L^2(0,T;V_A)$ as $\ell \to \infty$.

If $\phi_B : V_B \to \mathbb{R}$ is not convex then (4.8) together with Assumption AB implies

$$\begin{aligned}
\frac{1}{2}\|z_\ell(t)\|_A^2 &\leq \lambda \int_0^t \|u_\ell(s) - u(s)\|_A^2 ds + \int_0^t I_\ell(s)ds\\
&\leq c\lambda \int_0^t \|u_\ell(s) - \hat{u}_\ell(t)\|_A^2 ds + c\lambda \int_0^t \|z_\ell(s)\|_A^2 ds\\
&\quad + c\lambda \int_0^t \|Q_{m_\ell}(u(t) - u_0) - (u(t) - u_0)\|_A^2 ds + c\lambda \int_0^t \|u_\ell^0 - u_0\|_A^2 ds + \int_0^t I_\ell(s)ds\\
&\leq c\lambda \|u_\ell - \hat{u}_\ell\|_{L^2(0,T;V_A)}^2 + c\lambda \|Q_{m_\ell}(u - u_0) - (u - u_0)\|_{L^2(0,T;V_A)}^2 + c\lambda T\|u_\ell^0 - u_0\|_A^2\\
&\quad + c\lambda \int_0^t \|z_\ell(s)\|_A^2 ds + \int_0^t I_\ell(s)ds\\
&=: r_\ell + c\lambda \int_0^t \|z_\ell(s)\|_A^2 ds + \int_0^t I_\ell(s)ds
\end{aligned}$$

instead of (4.9). With Gronwall's lemma, we come up with

$$\frac{1}{2}\|z_\ell(t)\|_A^2 \leq \mathrm{e}^{2c\lambda t} r_\ell + \int_0^t I_\ell(s)ds + 2c\lambda \int_0^t \mathrm{e}^{2c\lambda(t-s)} \int_0^s I_\ell(\xi)d\xi ds.$$

We have that $r_\ell \to 0$ as $\ell \to \infty$. However, the crucial point here is the additional assumption that $u_\ell^0 \to u_0$ in $V_A$ as $\ell \to \infty$. This together with $\int_0^t I_\ell(s)ds \to 0$ as $\ell \to \infty$, the uniform boundedness of $\max_{t\in[0,T]}\left|\int_0^t I_\ell(s)ds\right|$ and Lebesgue's theorem on dominated convergence finally proves the assertion. □

Finally, we can identify $b$ with $Bu$ by a variant of Minty's monotonicity trick.

*Proof of Theorem 4.2.* We again write $\ell$ instead of $\ell'$. Let us start with the convex case such that $B : V_B \to V_B^*$ is monotone.

Let $z \in L^\infty(0, T; V_B)$ be arbitrary. From the fully discrete problem (4.1), we get

$$\int_0^T \langle f_\ell(t) - \hat{v}'_\ell(t) - Av_\ell(t), u_\ell(t) - u_\ell^0 \rangle dt + \int_0^T \langle Bu_\ell(t), u_\ell^0 \rangle dt = \int_0^T \langle Bu_\ell(t), u_\ell(t) \rangle dt$$
$$= \int_0^T \langle Bu_\ell(t) - Bz(t), u_\ell(t) - z(t) \rangle dt + \int_0^T \langle Bu_\ell(t), z(t) \rangle dt + \int_0^T \langle Bz(t), u_\ell(t) - z(t) \rangle dt$$
$$\geq \int_0^T \langle Bu_\ell(t), z(t) \rangle dt + \int_0^T \langle Bz(t), u_\ell(t) - z(t) \rangle dt. \tag{4.10}$$

In view of Lemma 4.3, 4.4, 4.5 and 4.6, we find

$$\int_0^T \langle b(t), u(t) \rangle dt = \int_0^T \langle f(t) - u''(t) - Au'(t), u(t) - u_0 \rangle dt + \int_0^T \langle b(t), u_0 \rangle dt$$
$$\geq \int_0^T \langle b(t), z(t) \rangle dt + \int_0^T \langle Bz(t), u(t) - z(t) \rangle dt$$

when passing to the limit as $\ell \to \infty$. In particular, we made use of the weak convergence $Av_\ell \rightharpoonup Au'$ in $L^2(0, T; V_A^*)$ together with the strong convergence $u_\ell - u_\ell^0 \to u - u_0$ in $L^2(0, T; V_A)$ as $\ell \to \infty$. We emphasize that $u$ and $u_0$ need not to take values in $V_A$.

Taking $z = u \pm \theta w$ for arbitrary $w \in L^\infty(0, T; V_B)$ and $\theta \in (0, 1]$, we thus obtain

$$\pm \int_0^T \langle b(t), w(t) \rangle dt \leq \pm \int_0^T \langle B(u(t) \pm \theta w(t)), w(t) \rangle dt. \tag{4.11}$$

The hemicontinuity of $B : V_B \to V_B^*$ together with the boundedness of $B : V_B \to V_B^*$ (and thus of $B : L^\infty(0, T; V_B) \to L^\infty(0, T; V_B^*)$) and Lebesgue's theorem on dominated convergence implies $b = Bu$ as $\theta \to 0$.

If $\phi_B : V_B \to \mathbb{R}$ is not convex then Assumption AB leads to

$$\int_0^T \langle f_\ell(t) - \hat{v}'_\ell(t) - Av_\ell(t), u_\ell(t) \rangle dt$$
$$\geq -\lambda \int_0^T \|u_\ell(t) - z(t)\|_A^2 dt + \int_0^T \langle Bu_\ell(t), z(t) \rangle dt + \int_0^T \langle Bz(t), u_\ell(t) - z(t) \rangle dt$$

instead of (4.10), where we now take $z = u \pm \theta w$ for $w \in L^\infty(0, T; V)$.

Recall that $u \in L^2(0, T; V_A)$ under the additional assumption that $u_\ell^0 \to u_0$ in $V_A$ as $\ell \to \infty$. Employing the strong convergence $u_\ell \to u$ in $L^2(0, T; V_A)$ as $\ell \to \infty$ (see Lemma 4.6) shows that

$$\int_0^T \|u_\ell(t) - z(t)\|_A^2 dt \to \theta^2 \int_0^T \|w(t)\|_A^2 dt,$$

and we come up with

$$\pm \int_0^T \langle b(t), w(t) \rangle dt \leq \theta\lambda \int_0^T \|w(t)\|_A^2 dt \pm \int_0^T \langle B(u(t) \pm \theta w(t)), w(t) \rangle dt$$

instead of (4.11), from which we again conclude that $b = Bu$ as $\theta \to 0$. □

## 5. Examples

We will now consider the specific examples mentioned in the introduction in sufficient detail to demonstrate that Theorems 4.2 applies. In other words, we will verify Assumptions A, B, AB, P and IC thereby obtaining existence of solutions as well as a strongly convergent numerical method for approximating a solution.

In what follows, $\Omega$ always denotes an open bounded subset of $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$.

### *5.1. Martensitic transformations in shape memory alloys*

Consider (1.2) in $\Omega \times (0, T)$ supplemented by homogeneous Dirichlet boundary conditions for $u$ as well as initial conditions for $u$ and $u_t$. Assume that the continuous function $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ fulfills (2.5) for some $\lambda \geq 0$ and is the derivative of some $\varphi : \mathbb{R}^d \to \mathbb{R}$, where $\varphi$ is, for example, a double-well potential. Assume further that there exist $p > 1$ and $\mu, c_1, c_2 \geq 0$ such that for all $x \in \mathbb{R}^d$

$$\varphi(x) \geq \mu\, |x|^p - c_1 \quad \text{and} \quad |\sigma(x)| \leq c_2 (1 + |x|)^{p-1} . \tag{5.1}$$

Note that these assumptions on $\sigma$ and $\varphi$ are the simplest in order to show that the corresponding operator $B$ satisfies Assumption B.

To obtain a generalized formulation in the form (1.1), we choose $V_A = H_0^1(\Omega)$, $V_B = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ (using the standard notation for Lebesgue and Sobolev spaces). All the required assumptions on the function spaces are fulfilled and, in particular, $V_A$ is compactly embedded in $H$ because of Rellich's theorem.

We define the operators $A : V_A \to V_A^*$ and $B : V_B \to V_B^*$ via

$$\langle Aw, z\rangle = \int_\Omega \nabla w \cdot \nabla z\, dx\,, \quad w, z \in V_A\,, \quad \langle Bw, z\rangle = \int_\Omega \sigma(\nabla w) \cdot \nabla z\, dx\,, \quad w, z \in V_B.$$

The potential $\phi_B : V_B \to \mathbb{R}$ is given by

$$\phi_B(w) := \int_\Omega \varphi(\nabla w)\, dx\,, \quad w \in V_B.$$

Then Assumptions A, B and AB are fulfilled. In particular, we observe that for all $w, z \in V$

$$\langle Bw - Bz, w - z\rangle \geq -\lambda \int_\Omega |\nabla w - \nabla z|^2 dx = -\lambda\, \|w - z\|^2_{H_0^1(\Omega)}\, .$$

Finally, Assumption P can be satisfied by using suitable finite element spaces, see Boman [6] as well as Crouzeix and Thomée [13]. Assumption IC is satisfied for suitable initial data.

Hence, due to Theorem 4.2, there is a weak solution to this problem.

### *5.2. An example with $H^{-1}(\Omega)$ as the pivot space*

Consider (1.3) in $\Omega \times (0, T)$ supplemented by the boundary condition $\sigma(u) = 0$ on $\partial\Omega \times (0, T)$ and by initial conditions for $u$ and $u_t$. Assume that the continuous function $\sigma : \mathbb{R} \to \mathbb{R}$ is given by $\sigma = \varphi'$ for some $\varphi : \mathbb{R} \to \mathbb{R}$ and fulfills (2.5) and (5.1) for all $x \in \mathbb{R}$, where $1 < p < \infty$ if $d \in \{1, 2\}$ and $2d/(d + 2) \leq p < \infty$ if $d \geq 3$.

Let $V_A = L^2(\Omega)$, $V_B = L^p(\Omega)$ and $H = H^{-1}(\Omega)$. Then all the required assumptions on the function spaces are fulfilled. The use of $H^{-1}(\Omega)$ as the pivot space has been considered, in particular, in Lions [30, pp. 191f.] and Gajewski, Gröger and Zacharias [27, pp. 72f.]. For the study of the full discretization of nonlinear evolution equations of first order with $H^{-1}(\Omega)$ as the pivot space, we also refer to Emmrich and Šiška [19].

The operator $A : V_A \to V_A^*$ is just the identity (and so Assumption A is trivially satisfied), while $B : V_B \to V_B^*$ and $\phi_B : V_B \to \mathbb{R}$ are defined via

$$\langle Bw, z\rangle = \int_\Omega \sigma(w) z \, dx \,, \quad \phi_B(w) := \int_\Omega \varphi(w) \, dx \,, \quad w, z \in V_B \,.$$

Also Assumptions B and AB then are satisfied.

Assumption IC can be satisfied by a suitable choice of the initial data.

In order to satisfy Assumption P, we need to show the stability of the $H$-orthogonal projections $P_m : H \to V_m$ with respect to $V = V_A \cap V_B$. In the one dimensional case, this has been shown in Emmrich and Šiška [19, Section 4], where $V_m$ consists of piecewise constant functions.

Then, due to Theorem 4.2, there is a weak solution to this problem.

It is perhaps interesting to note that, in the one dimensional case, equation (1.2) is formally equivalent to

$$\begin{cases} w_t - v_x = 0, \\ v_t - v_{xx} - \sigma(w)_x = f. \end{cases} \tag{5.2}$$

This can be seen by taking $v = u_t$ and $w = u_x$. This problem is studied, for example, in Dressel and Rohde [18].

Furthermore, taking the derivative with respect to $t$ in the first equation in (5.2) and the derivative with respect to $x$ in the second one, we formally arrive at

$$w_{tt} - (w_t)_{xx} - \sigma(w)_{xx} = f_x,$$

which is exactly of type (1.3).

### 5.3. *An equation with no spatial derivatives on the zero order term*

In this example, the fractional Laplacian is applied to the first-order-in-time term. For $s \in (1/2, 1]$, consider equation (1.4) supplemented by homogeneous boundary conditions for $u$ and initial conditions for $u$ and $u_t$.

Assume that $\sigma : \mathbb{R} \to \mathbb{R}$ is given by $\sigma = \varphi'$ for some $\varphi : \mathbb{R} \to \mathbb{R}$ and that again (2.5) and (5.1) are satisfied for all $x \in \mathbb{R}$.

Let $V_A = H_0^s(\Omega)$ be the standard Sobolev–Slobodetskii space (see, e.g., McLean [33]), $V_B = L^p(\Omega)$ and $H = L^2(\Omega)$. Then all the required assumptions are fulfilled. Note that $H_0^s(\Omega)$ is compactly embedded in $L^2(\Omega)$ for $s > 0$ (see, e.g., McLean [33, Theorem 3.27]).

The operator $A : V_A \to V_A^*$ is defined via

$$\langle Aw, z\rangle = \frac{1}{2} c_{d,s} \iint_{\Omega\times\Omega} \frac{(w(y) - w(x))(z(y) - z(x))}{|y - x|^{d+2s}} \, dxdy \,,$$

where $c_{d,s} = \pi^{-d/2} s 4^s \Gamma((d+2s)/2)/\Gamma(1-s)$, and satisfies Assumption A because of the Friedrichs inequality. The operator $B : V_B \to V_B^*$ and the potential $\phi_B : V_B \to \mathbb{R}$ are defined as in the previous example. Again, Assumptions B and AB are satisfied. In particular, we observe that for all $w, z \in V$

$$\langle Bw - Bz, w - z\rangle \geq -\lambda \int_\Omega |w - z|^2 dx = -\lambda \|w - z\|^2_{L^2(\Omega)} \geq -c\lambda \|w - z\|^2_{H^s_0(\Omega)}.$$

Assumption P can be satisfied, e.g., in view of the results in Boman [6], Crouzeix and Thomée [13] and Steinbach [42]. As in the previous examples, Assumption IC can be satisfied for a suitable choice of initial data.

Hence, due to Theorem 4.2, there is a weak solution also to this problem.

We should mention that other definitions of the fractional Laplacian may be considered. The definition above corresponds to the so-called regional fractional Laplacian (see, e.g., Guan and Ma [29]). Moreover, one may study the case $0 < s < 1/2$. Then, however, the boundary condition does not make sense and the Friedrichs inequality is not at hand, so that $(-\Delta)^s u$ should be replaced by $(-\Delta)^s u + u$ in order to have a strongly positive operator.

## Appendix: An integration-by-parts formula

In what follows, let $X$ and $Y$ be real, reflexive and separable Banach spaces and $H$ be a Hilbert space such that

$$X \subseteq Y \subseteq H = H^* \subseteq Y^* \subseteq X^*$$

holds with dense and continuous embeddings. Let $p, q \in (1, \infty)$ with $p^*$, $q^*$ denoting the conjugate exponents.

**Lemma 5.1.** *Let $a \in L^p(0, T; X)$ with $a' \in L^{q^*}(0, T; Y^*)$. Let $b \in L^q(0, T; Y) \cap L^\infty(0, T; H)$ with $b' \in L^{p^*}(0, T; X^*)$. If $p \geq q$ then $a \in \mathscr{C}([0, T]; H)$, $b \in \mathscr{C}_w([0, T]; H)$, and there holds for all $\alpha, \beta \in [0, T]$*

$$\int_\alpha^\beta \langle b'(s), a(s)\rangle ds = (a(\beta), b(\beta)) - (a(\alpha), b(\alpha)) - \int_\alpha^\beta \langle a'(s), b(s)\rangle ds. \tag{5.3}$$

*Proof.* We consider the Banach spaces

$$\begin{aligned}
\mathscr{X} &:= \{w \in L^p(0, T; X) : w' \in L^{q^*}(0, T; Y^*)\},\ \|w\|_{\mathscr{X}} := \|w\|_{L^p(0,T;X)} + \|w'\|_{L^{q^*}(0,T;Y^*)},\\
\mathscr{Y} &:= \{w \in L^q(0, T; Y) \cap L^\infty(0, T; H) : w' \in L^{p^*}(0, T; X^*)\},\\
&\qquad \|w\|_{\mathscr{Y}} := \|w\|_{L^q(0,T;Y)} + \|w\|_{L^\infty(0,T;H)} + \|w'\|_{L^{p^*}(0,T;X^*)}.
\end{aligned}$$

In what follows, we outline the construction of sequences of sufficiently smooth functions approximating $a \in \mathscr{X}$ and $b \in \mathscr{Y}$ (focusing only on the approximation of $b$).

Let $\{\psi_0, \psi_1, \psi_2\}$ be a smooth partition of unity subordinate to the open cover of $[0, T]$ by the intervals $(-2H, 2H), (H, T - H), (T - 2H, T + 2H)$, where $H \in (0, T/4)$ is arbitrary but fixed. Let $w_j = \psi_j b$ ($j = 0, 1, 2$). We then find $b = w_0 + w_1 + w_2$ on $[0, T]$. Because of $w_j' = \psi_j' b + \psi_j b' \in L^\infty(0, T; H) + L^{p^*}(0, T; X^*) \subseteq L^{p^*}(0, T; X^*)$, we have $w_j \in \mathscr{Y}$ ($j = 0, 1, 2$).

For sufficiently small $h > 0$, let

$$w_{0h}(t) = \begin{cases} w_0(t + h) & \text{for } -h \leq t \leq T - h\,, \\ 0 & \text{for } t > T - h\,. \end{cases}$$

The continuity of the translation in Bochner–Lebesgue spaces with finite Lebesgue exponent (see, e.g., Gajewski, Gröger and Zacharias [27, Kapitel IV, Lemma 1.5]) implies, as $h \to 0$,

$$w_{0h} \to w_0 \text{ in } L^q(0,T;Y)\,, \quad w_{0h}' \to w_0' \text{ in } L^{p^*}(0,T;X^*)\,.$$

Moreover, one can easily show that, as $h \to 0$,

$$w_{0h} \overset{*}{\rightharpoonup} w_0 \text{ in } L^\infty(0,T;H)$$

by employing, in particular, the continuity of the translation in $L^1(0,T;H)$. Let $\{\rho_\varepsilon\}$ be a sequence of mollifiers with sufficiently small support. The continuity of the translation implies the continuity of the mollification, and we find, as $\varepsilon \to 0$,

$$\rho_\varepsilon * \overline{w_{0h}} \to w_{0h} \text{ in } L^q(0,T;Y)\,, \quad \rho_\varepsilon * \overline{w_{0h}'} \to w_{0h}' \text{ in } L^{p^*}(0,T;X^*)\,,$$

where the bar denotes extension by zero outside $[-h,T]$. One can also show that, as $\varepsilon \to 0$,

$$\rho_\varepsilon * \overline{w_{0h}} \overset{*}{\rightharpoonup} w_{0h} \text{ in } L^\infty(0,T;H)$$

by employing the continuity of the mollification in $L^1(0,T;H)$. Finally, we find

$$(\rho_\varepsilon * \overline{w_{0h}})' = \rho_\varepsilon * \overline{w_{0h}'} \quad \text{on } (0,T)\,.$$

The functions $w_1$ and $w_2$ can be dealt with similarly. By this construction, we obtain a sequence $\{b_k\} \subset \mathscr{C}^1([0,T];Y)$ such that, as $k \to \infty$,

$$b_k \to b \text{ in } L^q(0,T;Y)\,, \quad b_k \overset{*}{\rightharpoonup} b \text{ in } L^\infty(0,T;H)\,, \quad b_k' \to b' \text{ in } L^{p^*}(0,T;X^*)\,.$$

Analogously, we can construct a sequence $\{a_k\} \subset \mathscr{C}^1([0,T];X)$ such that, as $k \to \infty$,

$$a_k \to a \text{ in } L^p(0,T;X)\,, \quad a_k' \to a' \text{ in } L^{q^*}(0,T;Y^*)\,.$$

Since $L^p(0,T;X)$ is continuously embedded in $L^q(0,T;Y)$, we have $a \in \mathscr{X} \hookrightarrow \mathscr{C}([0,T];H)$ (see, e.g., Roubíček [41, Lemma 7.3]), and, for any $t \in [0,T]$, the trace operator $\Gamma_t^{\mathscr{X}} : \mathscr{X} \to H$, $\Gamma_t^{\mathscr{X}} w = w(t)$, is linear and bounded and thus continuous. In particular, we have that $a_k(\alpha) \to a(\alpha)$ and $a_k(\beta) \to a(\beta)$ in $H$ as $k \to \infty$.

We further observe that $b \in \mathscr{Y} \subseteq L^\infty(0,T;H) \cap \mathscr{AC}([0,T];X^*) \subseteq \mathscr{C}_w([0,T];H)$ (see, e.g., Lions and Magenes [31, Chapitre 3, Lemme 8.1]). Moreover, one can show that the trace operator $\Gamma_t^{\mathscr{Y}} : \mathscr{Y} \to H$, $\Gamma_t^{\mathscr{Y}} w = w(t)$, is linear and demicontinuous. As a mapping of $\mathscr{Y}$ into $X^*$, the trace operator $\Gamma_t^{\mathscr{Y}}$ is linear and bounded. We thus have $b_k(\alpha) \to b(\alpha)$ and $b_k(\beta) \to b(\beta)$ in $X^*$ as $k \to \infty$. On the other hand, the sequences $\{b_k(\alpha)\}$ and $\{b_k(\beta)\}$ are bounded in $H$. Therefore, there exists a subsequence, denoted by $k'$, such that $\{b_{k'}(\alpha)\}$ and $\{b_{k'}(\beta)\}$ are weakly convergent in $H$. Because of the strong convergence in $X^*$, the limit, however, can only be $b(\alpha)$ and $b(\beta)$, respectively. By contradiction, one can then show that indeed the whole sequence $\{b_k(\alpha)\}$ and $\{b_k(\beta)\}$ converges weakly in $H$ towards $b(\alpha)$ and $b(\beta)$, respectively.

For $a_k, b_k$, we can now carry out integration by parts and obtain

$$\int_\alpha^\beta \langle b_k'(s), a_k(s)\rangle ds = (a_k(\beta), b_k(\beta)) - (a_k(\alpha), b_k(\alpha)) - \int_\alpha^\beta \langle a_k'(s), b_k(s)\rangle ds.$$

Passing to the limit proves the assertion. Note that all the terms appearing in (5.3) are well-defined. □

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